ON THE GEOMETRIC ANALYSIS
AND ADJUSTMENT
OF OPTICAL SATELLITE OBSERVATIONS

by

Emmanuel Tsimis

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PREFACE

This project is under the supervision of Ivan I. Mueller, Professor of the Department of Geodetic Science at The Ohio State University, and it is under the technical direction of Benjamin Milwitzky, Project Manager, Geodetic Satellite Program, NASA Headquarters, Washington, D. C. The contract is administered by the Office of University Affairs, NASA, Washington, D. C. 20546.

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1. INTRODUCTION

The science of geodesy has found in artificial satellites new fruitful means toward accomplishing its objectives. Thus the new branch of satellite geodesy was born, comprising the methods of treatment of the geodetic problems which employ the new celestial bodies.

Satellite geodesy methods may be considered as falling in three categories:

(a) geometric methods, employing mostly geometric principles,

(b) dynamic methods, whose results rely mostly on dynamic principles, and

(c) mixed methods, when no discernment can be drawn between the previous two.

The above categories furnish the basis for the distinction between geometric satellite geodesy and dynamic satellite geodesy.

Given a science, say geodesy, its objectives are more or less defined and what changes are the methods and the means scientists invent and employ to come closer to the solutions of the problems in question. Therefore, comparison of the results from different methods and means employed for the same objective is made very often.

It seems reasonable to state that the more independent the employed methods and means are from each other, the more reliable are the inferences from the comparison of the corresponding results.

Considering the case of geometric and dynamic satellite geodesy, there appear to be problems which may be treated with methods of either one of the two branches. One example is the determination of the equatorial radius of the earth.

In this and analogous cases, one branch serves the other better, when the results rely as much as possible upon the principles of the branch within which they were obtained. Consequently geometric satellite geodesy problems should

be treated in such a way that dynamic principles involvement is avoided as much as possible.

It is illuminating to trace back to the fundamental principles upon which the geometric satellite geodesy is founded.

The bodies of the earth and the satellite are considered in continuous motion in space. Their motions are not independent from each other, but nevertheless we do not know the exact relationship between them because of disturbing factors of not well known effects. Geometric satellite geodesy wants to ignore any knowledge of the relative motion of the satellite with respect to the earth, and considers the satellite at certain discrete moments as a target point in space. Instead of the bodies of the earth and the satellite being in continuous motion, now we have only the solid earth in continuous motion and a set of discrete points Q_1, Q_2, ... Q_n corresponding to the satellite positions at times t_1, t_2, ... t_n.

One should look at the problem as following. At time t_i the satellite is at the point Q_i while the solid earth has a unique relative position E_i with respect to Q_i. This is an event which is given the designation $\lceil E_i, Q_i/t_i \rceil$.

If there exists a means to snap, so to say, an event $\lceil E_i, Q_i/t_i \rceil$ in such a way that the relative position of Q_i with respect to the solid earth may be recovered, then one says that the point Q_i may be tied to the solid earth. Having the points Q_1, Q_2, ... Q_n tied to the solid earth we may consider the solid which comprises the solid earth and the finite set of points Q_1, Q_2, ... Q_n. Let this solid be called <u>extended field</u> of geometric satellite geodesy. The field of the earth-tied geometric geodesy is defined on the set of points of the solid earth with operations the angle and distance measurements. The field of geometric satellite geodesy is an extension, so to say, of the field of the earth-tied geometric geodesy with the same operations, i.e., the angle and distance measurements. Problems which could not be solved in the field of the earth-tied geometric geodesy are solved in the field of geometric satellite geodesy. This is a point where one is tempted to recall in mind the analogy with the algebraic fields. No matter how much apart in nature the two

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cases are, one cannot deny the analogy of the above geodetic case with that of the fields of real and complex numbers. For example, the equation $x^2 + 1 = 0$ does not have any solution in the field of real numbers, but it does have in the field of complex numbers, which is obtained from the former after the adjunction of the element $i = \sqrt{-1}$. If nothing else this analogy hints some beauty in formulation, deeper understanding and broad perception of the situation, gained when the concrete structures of the specific problem in question are identified with the corresponding abstract mathematical structures.

A method of solution of a problem in geometric satellite geodesy might be considered, just for description purposes, as anticipating two things: (1) the recovery of the relative positions of the extension-points with respect to the solid earth, and (2) the solution of the problem itself. These two things are done together in a unified way; however, this remark is of importance for the formulation of the problem.

The recovery of the extension point Q_i of the event $\ulcorner E_i, \ Q_i/t_i \urcorner$ is achieved by performing simultaneous observations in a certain proper mode from a number of ground stations. Depending upon the observational mode and the problem to be solved, there exists a minimum for the number of participating stations at each event, and the number of events needed for a unique solution of the problem.

Geometric satellite geodesy has at its disposition today three basically different observational modes:

        (a) the optical observations mode,

        (b) the range observations mode, and

        (c) the range difference observations mode.

The problems of geometric satellite geodesy usually consist in determining the cartesian coordinates of a set of ground stations. It might happen some of them to be of known position. In this case the problem is easier than the problem where all the stations are of unkown position. Geometric satellite geodesy methods provide solutions to the general problem where all the stations are considered of unknown position. Geometrically speaking all the observational

modes are equivalent to each other with respect to that problem.

In the optical observations mode each participating station P_i at an event $\lceil E_i \, Q_i / t_i \rceil$ observes quantities that are geometrically equivalent to the directional cosines of the direction $\overrightarrow{P_i Q_i}$ with respect to a coordinate system which must be connected with the solid earth. It should be explained here that the coordinate system is considered connected with the solid earth either when it is tied to the solid earth, or when it is not tied but the relative motion is considered known. Here and throughout this work the employed coordinate system is considered tied to the solid earth.

The problem to be solved is the determination of the cartesian coordinates of the participating stations. The questions to be answered in the following sections are:

(a) What information do the observations alone (without the introduction of additional data) provide about the configuration of the stations and what are actually the quantities to be adjusted?

(b) What additional data are needed to be introduced in order to determine the coordinates of the stations? Are these additional elements necessary during the adjustment?

2. THE CASE OF TWO OBSERVING STATIONS

2.1 Geometric Analysis

As it has been stressed earlier each method of geometric satellite geodesy must anticipate observations for the recovery of the extension points (satellite position points) on the one hand and the estimation of the unknowns of the problem on the other.

Since two intersecting directions determine a point it is concluded that in the optical observations mode the minimum number of participating stations at each event is two (at least for the tying of the extension points Q_j). It will be examined presently whether that information at each event is enough to solve the posed problem.

Fig. 1

Let P_1 and P_2 be two ground stations which observe simultaneously at the instances $\lceil t_j \rceil = \lceil t_1, t_2, \ldots, t_p \rceil$ the satellite which at these instances is at the points $\lceil Q_j \rceil = \lceil Q_1, Q_2, \ldots, Q_p \rceil$ respectively.

Fig. 2

At the instant t_j the stations P_1 and P_2 observe the point Q_j and obtain information equivalent to the directional cosines $\lceil a_{1j}, b_{1j}, c_{1j} \rceil$ of the direction $P_1 Q_j$

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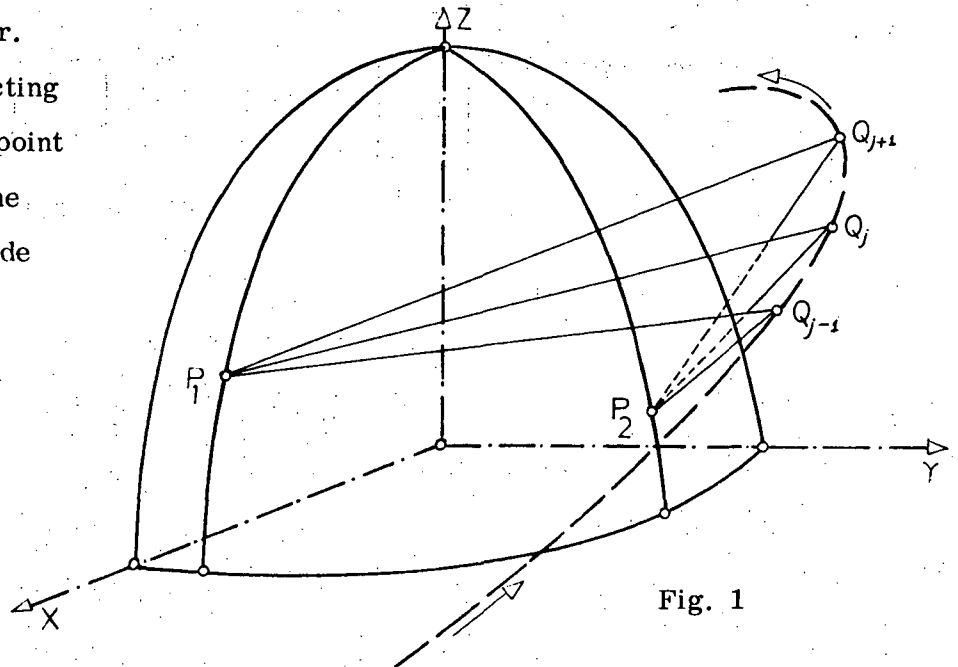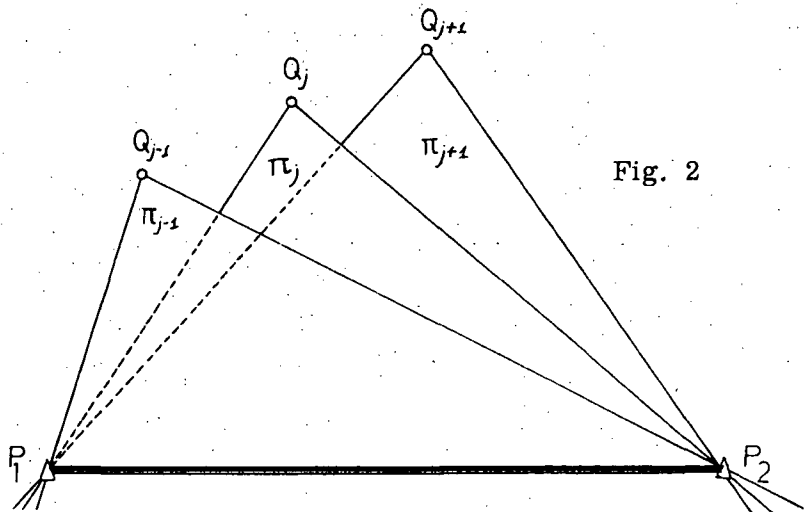
and $\lceil a_{2j}, b_{2j}, c_{2j} \rceil$ of the direction P_2Q_j, with respect to some cartesian coordinate system defined above.

Let the following notation stand for the adjacent discription:

X_j, Y_j, Z_j Coordinates of Q_j

x_i, y_i, z_i Coordinates of ground station P_i

$\lceil a_{1j}, b_{1j}, c_{1j} \rceil = \overline{m}_i$ Directional cosines vector of the direction P_iQ_j, where a, b, c correspond to the x, y, z-axes respectively.

π_j Plane defined by Q_j and the two observed directions.

There appear to be three distinct views, namely

    (a) the plane defined by the three points P_1, P_2, Q_j

    (b) the plane defined by the line P_1P_2 and the point Q_j

    (c) the plane defined by the intersecting directions P_1Q_j and P_2Q_j which is the above designated by π_j

The question arises as to which one of these fits the actual situation in this problem. It is not difficult to single out the last one. As a matter of fact there are two observed directions toward the point Q_j. Although the observations are made from P_1 and P_2 the station position is not specified on the respective directions. Thus one may not consider the points P_1, P_2 or the line P_1P_2 for the definition of the plane. There remains only π_j and let it be called <u>fundamental simplex</u> of the optical observation mode.

The equation of this fundamental simplex is that of a plane which passes from a point $Q_j\lceil X_j, Y_j, Z_j \rceil$ and is parallel to two vectors $\overline{m}_1 = \lceil a_{1j}, b_{1j}, c_{1j} \rceil$ and $\overline{m}_2 = \lceil a_{2j}, b_{2j}, c_{2j} \rceil$. That is

$$\pi_j : \quad \begin{vmatrix} x - X_j & y - Y_j & z - Z_j \\ a_{1j} & b_{1j} & c_{1j} \\ a_{2j} & b_{2j} & c_{2j} \end{vmatrix} = 0, \tag{1}$$

or in normal form

$$\alpha_j x + \beta_j y + \gamma_j z + \delta_j = 0, \tag{2}$$

where

$$\alpha_j = \begin{vmatrix} b_{1j} & c_{1j} \\ b_{2j} & c_{2j} \end{vmatrix},$$

(3)

$$\beta_j = \begin{vmatrix} c_{1j} & a_{1j} \\ c_{2j} & a_{3j} \end{vmatrix},$$

(4)

$$\gamma_j = \begin{vmatrix} a_{1j} & b_{1j} \\ a_{2j} & b_{2j} \end{vmatrix},$$

(5)

and

$$\delta_j = - \begin{vmatrix} X_j & Y_j & Z_j \\ a_{1j} & b_{1j} & c_{1j} \\ a_{2j} & b_{2j} & c_{2j} \end{vmatrix}.$$

(6)

Suppose that index j goes over the values $1, 2, \ldots, p$. Then there is a set of p planes π_j $(j = 1, 2, \ldots, p)$. Notice that each of these planes is defined independently from the others. Any pair of these planes uniquely defines an undirected line. There exist $C_2^p = p(p-1)/2$ such lines.

Consider any two from the planes π_{12}, say $j = j$ and $j = k$, i.e.,

$$\pi_j \quad : \quad \alpha_j x + \beta_j y + \gamma_j z + \delta_j = 0$$

(7)

and

$$\pi_j \quad : \quad \alpha_k x + \beta_k y + \gamma_k z + \delta_k = 0.$$

(8)

Let ι_{jk} be the line of intersection of these two planes and a_{jk}, b_{jk}, c_{jk} its direction cosines. Now the following theorem of analytic geometry will be proved.

<u>Theorem</u>   The direction cosines of the line of intersection of two intersecting planes are proportional to the two-rowed minors of the coefficient matrix of the equations of the two planes, taken alternately with the plus and the minus signs.

<u>Proof.</u> Consider any two points P_1 and P_2 of the line ι_{jk} of intersection. Then

$$\alpha_j x_1 + \beta_j y_1 + \gamma_j z_1 + \delta_j = 0, \qquad \alpha_k x_1 + \beta_k y_1 + \gamma_k z_1 + \delta_k = 0,$$

$$\alpha_j x_2 + \beta_j y_2 + \gamma_j z_2 + \delta_j = 0, \text{ and } \quad \alpha_k x_2 + \beta_k y_2 + \gamma_k z_2 + \delta_k = 0.$$

or

$$\alpha_j(x_1 - x_2) + \beta_j(y_1 - y_2) + \gamma_j(z_1 - z_2) = 0$$

and

$$\alpha_k(x_1 - x_2) + \beta_k(y_1 - y_2) + \gamma_k(z_1 - z_2) = 0.$$

But

$$\frac{x_1 - x_2}{a_{jk}} = \frac{y_1 - y_2}{b_{jk}} = \frac{z_1 - z_2}{c_{jk}}$$

and then

$$\alpha_j a_{jk} + \beta_j b_{jk} + \gamma_j c_{jk} = 0$$

$$\alpha_k a_{jk} + \beta_k b_{jk} + \gamma_k c_{jk} = 0$$

or

$$\begin{bmatrix} \alpha_j & \beta_j \\ \alpha_k & \beta_k \end{bmatrix} \begin{bmatrix} a_{jk} \\ b_{jk} \end{bmatrix} = -\begin{bmatrix} \gamma_j \\ \gamma_k \end{bmatrix} c_{jk}.$$

By Cramer's rule the last system yields:

$$a_{jk} = \frac{\begin{vmatrix} -\gamma_j & \beta_j \\ -\gamma_k & \beta_k \end{vmatrix}}{\begin{vmatrix} \gamma_j & \beta_j \\ \gamma_k & \beta_k \end{vmatrix}} c_{jk} \quad \text{and} \quad b_{jk} = \frac{\begin{vmatrix} \alpha_j & -\gamma_j \\ \alpha_k & -\gamma_k \end{vmatrix}}{\begin{vmatrix} \alpha_j & \beta_j \\ \alpha_k & \beta_k \end{vmatrix}} c_{jk}$$

or

$$\frac{a_{jk}}{\begin{vmatrix} \beta_j & \gamma_j \\ \beta_k & \gamma_k \end{vmatrix}} = \frac{b_{jk}}{\begin{vmatrix} \gamma_j & \alpha_j \\ \gamma_k & \alpha_k \end{vmatrix}} = \frac{c_{jk}}{\begin{vmatrix} \alpha_j & \beta_j \\ \alpha_k & \beta_k \end{vmatrix}} = \frac{1}{\lambda_{jk}} \qquad (9)$$

Q.E.D.

It is of interest to find the value of λ_{jk}.

Equation (9) yields:

$$1 = a_{jk}^2 + b_{jk}^2 + c_{jk}^2 = \frac{1}{\lambda_{jk}^2}\left[\begin{vmatrix} \beta_j & \gamma_j \\ \beta_k & \gamma_k \end{vmatrix}^2 + \begin{vmatrix} \gamma_j & \alpha_j \\ \gamma_k & \alpha_k \end{vmatrix}^2 + \begin{vmatrix} \alpha_j & \beta_j \\ \alpha_k & \beta_k \end{vmatrix}^2\right]$$

or

$$\lambda_{jk}^2 = \begin{vmatrix} \beta_j & \gamma_j \\ \beta_k & \gamma_k \end{vmatrix}^2 + \begin{vmatrix} \gamma_j & \alpha_j \\ \gamma_k & \alpha_k \end{vmatrix}^2 + \begin{vmatrix} \alpha_j & \beta_j \\ \alpha_k & \beta_k \end{vmatrix}^2 . \tag{10}$$

Therefore the direction cosines of ι_{jk} are completely defined from the two planes,
provided they intersect.

As it was mentioned before there exist $C_2^p = p(p-1)/2$ such intersection
lines ι_{jk}. The adjustment problem becomes obvious: What is the line ι best
fitting the $p(p-1)/2$ lines ι_{jk}? Statistically speaking, this is a regression
problem where all the components of the position vector of each "point" are
subject to error ([22] pp. 186-194).

The question springs up as to which quantity one should apply the least
squares optimum criterion in such a regression problem? There exists a uniquely
defined quantity between any two lines, namely their shortest distance which is
along their common perpendicular; but this is a length, and there exists no length
in the problem.

The observations are angular quantities that means absolute invariants for the
metric geometry and an adjustment of purely absolute invariants through relative
ones, as the "length" is, would be considered a spoiled one. It seems reasonable
to keep an adjustment as "pure" as possible. Therefore by principle the introduction
of any "length" is excluded here. The writer drafts here the principle of duality, and
prefers to go back enough into the fundamentals of geometry to found the adjust-
ment to be employed for this problem.

The following subsection is devoted to the application of that principle to the
problem in question, i.e., the best fitting line to a given set of lines. The result-
ing adjustment drawn through the duality principle will be called dual adjustment.

2.2 Dual Adjustment

2.21 The Duality Principle

A refresher introduction of the duality principle stressing the points that the problem in question needs more, is set forth now. Duality is not a theorem but concerns theorems. As any other principle in mathematics, duality principle belongs to the jurisdiction of "metamathematics." For wider comprehension of the concept, it will be given from two instructively different standpoints: (1) that of synthetic geometry which studies figures without employing formulas, and (2) that of analytic geometry which establishes a correspondence between the elements of a figure (whatever they are considered) and the elements of a set called coordinates, and studies geometry through formulas between the coordinates corresponding to geometric relations. The intrinsic discussion will be within the latter point of view.

Duality in Synthetic Geometry.

Each geometric figure may be considered as an assemblage of elements, i.e., "units" from which the figure may be constructed in some way. For example a plane-ellipse might be considered as the aggregate of points with which one of them moving along the ellipse succesively coincides. But equally well that plane-ellipse might be considered the envelope of the coplanar tangents at each of its points. Thus the same figure has been generated by using two different elements, i.e., the point and the unlimited line in the plane of the ellipse. This is the case for any plane figure. Analogously in space each figure may be considered as generated either from a moving point or from a moving plane. It is this double way of viewing a figure being generated on which the principle of geometric duality is based.

It was Poncelet, who first enunciated the principle of duality, which in the case of the plane is:

> Any theorem about properties of position of plane figures is accompanied by the so-called dual one, which has the words "point" and "line" mutually interchanged. For example the dual statement of "three points in a plane define a triangle", is "three lines in a plane define a triangle."

The duality principle in space is:

Any theorem about properties of position of figures in space is accompanied by its dual one which has the words "point" and "plane" mutually interchanged, while the word "line" in place. For example the dual statement of "three not <u>collinear points</u> define a plane," is "three not <u>coaxial planes</u> define a point."

In the above statements of duality there is no mention of the so-called mass relations, i.e., those which include distance angle, area, etc. In fact, the duality principle was introduced within the frame of projective geometry at a time when it was unknown how to deduce metric geometry from projective geometry.* The Cayley's principle that metric and affine geometry can be considered special cases of projective geometry made possible the extension of the duality principle over the mass-relations of metric geometry. This is the key for the solution of the problem in question. The analytic character of this problem naturally turns the discussion into the field of analytic geometry. However, the fact that the whole deal is in the three-dimensional geometric space, that means within visualization, helps somehow against unrealistic analytical representations.

While in synthetic geometry duality is based on the double way of viewing the geometrical figure being generated, in analytic geometry duality is based on the double interpretation of the formula that stands for a figure or its analytical representation with respect to a coordinate system introduced beforehand.

Duality in analytic geometry is extended in a unified way to spaces of any dimension. However, this discussion need not go beyond three-dimensional space. Some representative examples will now illustrate the principle. For the <u>duality in plane</u> consider the equation of a straight line in the plane with respect to a cartesian coordinate system, (orthogonal or oblique) i.e.,

$$ax + by + c = 0 \qquad\qquad (11)$$

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* The English geometer Cayley in 1859 presented a way to deduce affine and metric geometry from projective geometry.

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In homogeneous coordinates x_1, x_2, x_3 related to x and y through the relations

$$x = \frac{x_1}{x_3} \quad \text{and} \quad y = \frac{x_2}{x_3}, \quad (x_3 \neq 0)$$

equations (11) may be written as

$$ax_1 + bx_2 + cx_3 = 0. \tag{11\ensuremath{'}}$$

The last relation is a homogeneous linear equation with respect to x_1, x_2, x_3 and the same holds with respect to a, b, c. One could say that the relation is symmetric with respect to the triples $\{x_1,\ x_2,\ x_3\}$ and $\{a,\ b,\ c\}$.

Now to what extent are the a, b, c, conversely determined by the straight line? If one compares (11)$'$ with, say,

$$a'x_1 + b'x_2 + c'x_3 = 0 \tag{11\ensuremath{''}}$$

under the assumption that they represent the same straight line one has, considering the linear system (11)$'$ and (11)$''$, that *

$$\text{rank} \begin{bmatrix} a, & b, & c \\ a', & b', & c' \end{bmatrix} = 1$$

which implies

$$a = \lambda a', \quad b = \lambda b', \quad c = \lambda c'. \tag{12}$$

Consequently, the quantities a, b, c are determined by the straight line up to a common constant of proportionality.

Similarly each point determines x_1, x_2, x_3 up to a common factor of proportionality. Indeed, if $\{x_1',\ x_2',\ x_3'\}$ represents the same point with $\{x_1,\ x_2,\ x_3\}$ whose non-homogeneous coordinates are x and y then

$$x = \frac{x_1}{x_3} = \frac{x_1'}{x_3'}, \qquad y = \frac{x_2}{x_3} = \frac{x_2'}{x_3'}$$

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* Since $\lceil a,\ b,\ c \rceil$ and $\lceil a',\ b',\ c' \rceil$ are two non-vanishing linearly dependent vectors, each is a multiple of the other.

or equivalently if $x_3 = \mu x_3'$, then $x_1 = x \cdot x_3 = x \mu x_3' = \mu (x x_3') = \mu x_1'$ and

$x_2 = y \cdot x_3 = y \cdot \mu x_3' = \mu (y x_3') = \mu x_2'$, i.e.,

$$x_1 = \mu x_1', \quad x_2 = \mu x_2', \quad x_3 = \mu x_3'. \tag{13}$$

The important fact here is that one can associate coordinates to the straight line of the plane in the same way as with the points of the plane. Consequently each triple $\left[\xi, \eta, \zeta\right]$ may be interpreted either as a point or as a straight line of the reference plane. Now, all the theorems that state anything about the points of the reference plane, about the aggregate of such points, and about the relations among them are nothing but statements about triples $\left[\xi, \eta, \zeta\right]$, the aggregate of such triples and algebraic relations among them. But the very same triples may be interpreted either as points or lines of the reference plane.

Two statements, one being expressed in point coordinates and dealing with points, the other in straight line coordinates and dealing with straight lines correspond exactly to each other and are called <u>dual statements</u>. Analogously "point" and "line" are called <u>dual elements</u> in plane. As one realizes, dual statements are different interpretations of one and the same algebraic result.

In order to have a geometrical visualization of this example, that synthetic geometry would provide, one could proceed this way: Keep the straight line constant in the plane (i.e., $\frac{a}{b} = \alpha = $ constant and $\frac{c}{b} = \beta = $ constant) and let the point $\left(x = \frac{x_1}{x_3}, \quad y = \frac{x_2}{x_3} \right)$ varying. Then equation (11)′ represents the range of points of the line. By keeping the point $\left(x = \frac{x_1}{x_3}, \quad y = \frac{x_2}{x_3} \right)$ constant in the plane, and having the straight line $\left(\frac{b}{a} = \alpha, \quad \frac{c}{a} = \beta \right)$ varying the very same equation (11)′ represents the flat pencil of lines through the point $\left(x = \frac{x_1}{x_3}, \quad y = \frac{x_2}{x_3} \right)$.

For the <u>duality in three-dimensional space</u> consider the equation of a plane, say,

$$\alpha x + \beta y + \gamma z + \delta = 0, \tag{14}$$

or in homogeneous point coordinates x_1, x_2, x_3, x_4 related to the non-homogeneous x, y, z through the relations

$$x = \frac{x_1}{x_4}, \quad y = \frac{x_2}{x_4}, \quad z = \frac{x_3}{x_4}, \quad (x_4 \neq 0),$$

$$\alpha x_1 + \beta x_2 + \gamma x_3 + \delta x_4 = 0. \tag{14}'$$

Relation (14)' is a linear homogeneous equation with respect to x_1, x_2, x_3, x_4 and the same holds with respect to α, β, γ, δ.

In the following, the same steps which were followed for the equation of a straight line in a plane above are repeated. In fact, both of these cases are special cases of a general theory about hyperplane coordinates of a general projective space. Under the general theory, the commonly conceived planes are the "hyperplanes" of the three-dimensional projective space and the straight lines are the "hyperplanes" of the two-dimensional projective space. Nevertheless the same steps are repeated, for this is the space (i.e., the three-dimensional projective space) to which the problem in question belongs.

First, the question: to what extent are α, β, γ, δ, conversely determined by the plane? To answer this question equation (14)' is compared with, say,

$$\alpha' x_1 + \beta' x_2 + \gamma' x_3 + \delta' x_4 = 0 \tag{14}''$$

under the assumption that they represent the same plane. Then

$$\text{rank} \begin{bmatrix} \alpha & \beta & \gamma & \delta \\ \alpha', & \beta', & \gamma', & \delta' \end{bmatrix} = 1;$$

since the intersection of (14)' and (14)'' is just the plane itself, that means of two dimensions, and consequently these planes are linearly dependent. Therefore, there exists $\lambda \neq 0$ such that

$$\alpha = \lambda \alpha', \quad \beta = \lambda \beta', \quad \gamma = \lambda \gamma', \quad \delta = \lambda \delta'. \tag{15}$$

An alternative procedure is to write

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$$(\alpha x_1 + \beta x_2 + \gamma x_3 + \delta x_4) - \lambda(\alpha' x_1 + \beta' x_2 + \gamma' x_3 + \delta' x_4) = 0$$

or

$$(\alpha - \lambda \alpha')x_1 + (\beta - \lambda \beta')x_2 + (\gamma - \lambda \gamma')x_3 + (\delta - \lambda \delta')x_4 = 0, \tag{16}$$

for the linear dependence of the planes; then from the fact that (16) must be identically satisfied, (15) follows. Consequently, the α, β, γ, δ are determined by the plane up to a common constant of proportionality.

Similarly each point with non-homogeneous coordinates x, y, z determines x_1, x_2, x_3, x_4 up to a common constant of proportionality through the relations

$$x = \frac{x_1}{x_4}, \qquad y = \frac{x_2}{x_4}, \qquad z = \frac{x_3}{x_4}.$$

Indeed if x_1', x_2', x_3', x_4' represents the same point with x_1, x_2, x_3, x_4, then

$$x = \frac{x_1}{x_4} = \frac{x_1'}{x_4'}, \qquad y = \frac{x_2}{x_4} = \frac{x_2'}{x_4'}, \qquad z = \frac{x_3}{x_4} = \frac{x_3'}{x_4'},$$

or, by putting $x_4 = \mu x_4'$ it is obtained

$$x_1 = \mu x_1', \qquad x_2 = \mu x_2', \qquad x_3 = \mu x_3', \qquad x_4 = \mu x_4'. \tag{17}$$

Thus, one can associate coordinates to the plane in the same way as with the points in space.* Consequently each quadruple $\left[\xi, \eta, \zeta, \tau\right]$ may be interpreted either as a point or as a plane. This is the point where duality principle in space springs up. For, all the theorems that state anything about points in space, about aggregate of such points, and about relations among them are nothing but statements about homogeneous quadruples $\left[\xi, \eta, \zeta, \tau\right]$, aggregate of such quadruples, and algebraic relations among them. But the very same quadruples may be interpreted either as points or planes. Two statements, one being expressed in point coordinates and dealing with points, the other in plane coordinates and dealing with planes are called <u>dual statements.</u>

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* Here by space it is meant the three-dimensional projective space. The adjective projective has been dropped to avoid confusion for some readers.

Again one notices that, dual statements are different interpretations of one and the same algebraic result. The above example exhibited that "point" and "plane" are dual notions in space. No mention was made about the third fundamental element* of space, the straight line. But the dual statements

<div align="center">

"two not-coincident points define a line"

"two not-coincident planes define a line"

</div>

show clearly that "straight line" is a self-dual notion in space.

After the above general introduction of the duality principle, one has all that is necessary to formulate the problem in a way that turns out to be very fruitful. The following paragraph is devoted to this formulation and constitutes the key for the adjustment problem.

2.22 Formulation of the Problem through the Duality Principle

The geometric analysis of the optical observations mode from two observing stations (section 2.1) showed that the adjustment problem amounts to that of determining the best fitting straight line of a given set of straight lines in space. Each straight line of the set to be fitted is the intersection of a pair of planes from a finite set of planes. Therefore the adjustment problem may be stated: What is the best line (in some respect) through a set of planes? Instead of attacking this problem directly, it is easier to consider its dual one, which is a two-dimensional problem. The formulation of the dual problem is obtained by interchanging the words "point" and "plane" and leaving the word "line" in place. Thus the enunciation of the dual problem is: What is the best line (in some respect) through a set of points? This is a problem in two dimensions (see section 2.21), and its analysis occupies a first place in regression analysis. The optimum criterion for the "best" will be that of least squares. What is needed from this problem is the procedure of solution and not the solution itself.

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*The characterization fundamental element for "point", "plane" and "straight line", does not have any geometric justification. It is given on intuitive grounds only; for there are other "elements" with equal geometrical justification.

Therefore, the solution procedure of the two-dimensional dual problem will be sketched first, and then it will be translated through duality for the problem in question. The problem of fitting a straight line to a set of points in a plane depends upon how the coordinates of the points are treated in the adjustment, from a statistical point of view. Thus there exist two cases:

(i) Only one of the coordinates of each point (either x or y) may be subject to error, and

(ii) both of the coordinates may be subject to error.

In the first case, one obtains either the regression line of y's and x's (when only y's may be subject to error) or the regression line of x's on y's (when only x's may be subject to error).

In the second case, under some assumptions, there exists a unique regression line. The case of interest here is the second one. As it is very well known, the regression line in this case results from the minimum of the sum of the squares of the distances of the points from the line to be fitted, provided the two coordinates (x and y) are given equal weights*. If the coordinates x and y are given equal weights, then the line of the observed and adjusted point is perpendicular to the fitting line. It is supposed that this is the case here.
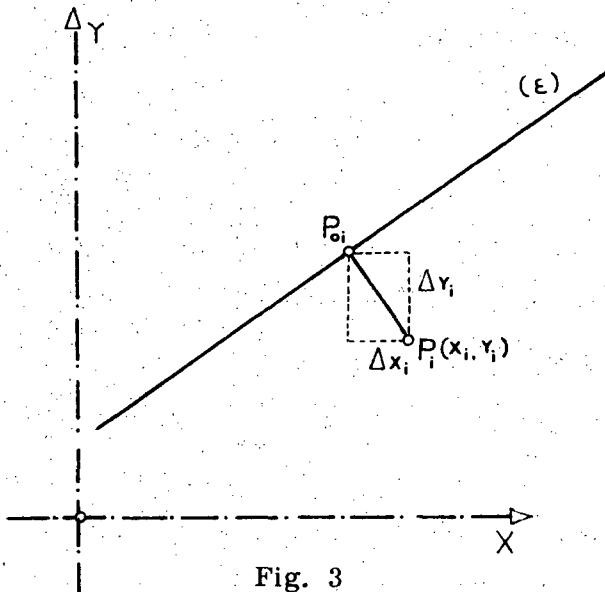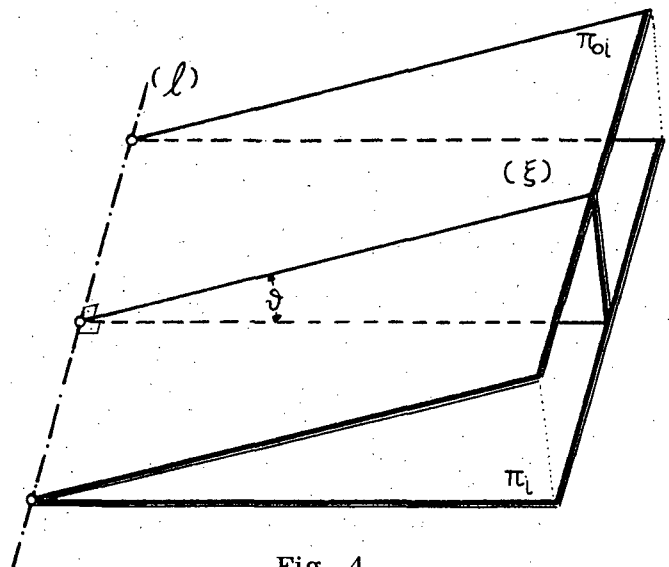

Fig. 3

Fig. 4

*If the coordinates x and y of a point P are not given equal weights, then the line connecting the observed and adjusted points is not perpendicular to the fitting line.

Let $P_i(x_i, y_i)$ be a set of points in a plane and (ϵ) the least-squares line through them (Fig. 3). If P_{oi} is the adjusted point of P_i, then $(P_i P_{oi}) \perp (\epsilon)$, and the least-squares criterion is $\sum_i d_i^2 = \min$.

Now let (π_i) be a set of planes in space, and (ξ) the least-squares line through them (Fig. 4). Then duality principle establishes the following correspondence between the elements of Figs. 3 and 4.

$$P_i \longleftrightarrow \pi_i$$
$$(\epsilon) \longleftrightarrow (\xi)$$
$$P_{oi} \longleftrightarrow \Pi_{oi}$$
$$(P_{oi} P_i) \longleftrightarrow (\lambda)$$
$$(\epsilon) \perp (P_{oi} P_i) \longleftrightarrow (\lambda) \perp (\xi).$$

Let $D(\pi_{oi}, \pi_i)$ be a symbol standing for a concept of "distance" between the planes π_{oi} and Π_i. Then the least-squares criterion $\sum_i d_i^2 = \min$. for points is translated into $\sum_i D^2(\pi_{oi}, \pi) = \min$. for planes. The question now arises as to what is the "distance" between two planes. One might think of the angle between the planes. At least intuitively angle is a suitable measure here, but there are many ways to measure a quantity, which are equivalent in some respect. Therefore, it is not hopeless to search for a suitable convenient measure of the "distance" between two planes. This is the target the following subsections aim at.

2.23 The Euclidean Geometry

The problem in question falls within the content of the three-dimensional Euclidean geometry. Although the Euclidean space is intuitively appealing, there exist some questions which are of importance not only theoretically, but also from the point of view of the applications.

It is not the purpose of this work to go over the foundation of the Euclidean geometry. However, trying to define an appropriate metric for the adjustment problem as set up above, it is inevitable to face the following question: What is considered a legitimate metric in Euclidean geometry and how is one to determine the totality of these metrics? This question does touch the foundation of Euclidean geometry. A rigorous presentation entails the parallel exposition of non-Euclidean geometries in the content of which Euclidean geometry is a special case. In order to reach the problem's goal without the employment of non-Euclidean geometries, for that would take the

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discussion too far, a conventional way will be followed, where rigorousness will be guaranteed by recalling theorems of the general theory.

In any system of geometry the existence of some fundamental elements, which constitute the objects of the operations of the geometry, is assumed. Different kinds of geometry arise considering different fundamental elements or by defining them in different ways. From this point of view in Euclidean geometry the existence of two objects is assumed:

(a) the generating element (either point, or straight line, or plane, or some other configuration of equal justification), and

(b) the distance between the generating elements.

Of course these are only the objects. In order to construct a geometry a certain set of axioms is needed upon which the structure of the geometry will be based. If now someone takes the point as the generating element and the distance between two points in the commonly known sense and adopts the set of axioms of Euclid, the point-Euclidean geometry is obtained. If instead of the point, one takes the straight line or the plane as the generating element, one obtains equally justified geometries: the line-Euclidean geometry or the plane-Euclidean geometry, respectively.

Geometries different from the Euclidean may be obtained either when the distance is defined in a different way, but equally justified (in the sense that a self-consistent geometric structure* may be built by using that), or when there exist different, equally justified sets of axioms. The latter case is that of non-Euclidean geometries, which arose by disputing Euclid's fifth axiom, the so-called parallel axiom**.

From the above point of view, geometry seems to possess no organic systematization. This, however, is not the case. Analytic representations in geometry appeared to be very fruitful and enabled geometers to develop geometry in a systematic way as a unified structure.

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*One may recall here as an example the Minkowskian geometry versus Euclidean (see [11], p. 133).

**Some geometers distinguish between axiom and postulate, the former being a simple fact of logic while the latter is about space perception.

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It was Klein's ingenious idea to introduce the notion of "motion" of figures in geometry which has provided the best systematization of geometry so far. Klein's view of geometry is simple, elegant and fruitful, and it will be adopted here.

In analytic geometry one represents the various geometric figures in terms of coordinates. For example, one thinks of the totality of points in the usual space as represented by the totality of triples x, y, z, i.e., by their coordinates with respect to some coordinate system. To every transformation of points in space there corresponds a certain transformation of their coordinates. The converse is not true. That is, not every analytic relation between coordinates corresponds to a geometric transformation. In other words, each system in geometry (or each "geometry" as it is usually said) possesses a well-defined group of motions, which can be represented analytically (in terms of coordinates). One defines then the geometry as the theory of the properties of the configurations which remain unchanged under those motions. In this light, Euclidean geometry is defined as the geometry which studies those relations between the coordinates, which remain unchanged by the linear substitutions of coordinates corresponding to the following motions:

> parallel displacements,
>
> rotations about the origin,
>
> reflections about the origin, and
>
> similarity transformations with the origin as center.

These substitutions are linear and constitute a group of transformations, the so-called principal group* of transformations. For a figure to remain unchanged under these "motions" means that it is "moved" as a solid in space; that in turn implies distance and angle preservation.

In the principal group, the motions involve six parameters, to which one must add one more parameter for the change in unit length (scale), so that altogether

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*This set of transformations constitute a group, for (i) the product of any two of them belongs to that set, (ii) it comprises the identity transformation and each transformation has its inverse one.

the principal group needs seven parameters to be defined completely.
Let this group be designated by G_7.

Geodesists are very familiar with this group. In each geometric adjustment in geodesy, the conditions and the constraints involved are dictated by the principal group as it is considered in connection with the geometrical configuration and the coordinate system of that adjustment.

2.24 The Euclidean Metric

Consider the three-dimensional geometric space, * as the set either of its points or its straight lines, or its planes. Let E_3^p, E_3^L, E_3^π be the designations for these three sets respectively. These are the three sets of interest of the problem in question. As it was shown earlier (while discussing the duality principle) one may introduce a coordinate system, and establish a correspondence between the elements of the sets E_3^p, E_3^L, E_3^π and the coordinates. Specifically there was established the correspondence of

points with the quadruples $\lceil x_1, x_2, x_3, x_4 \rceil$,

planes with the quadruples $\lceil \alpha, \beta, \gamma, \delta \rceil$, and

lines with the pairs of points or planes.

It was said also, that each point determines the quadruple $\lceil x_1, x_2, x_3, x_4 \rceil$ up to a common constant of proportionality, and similarly each plane determines the quadruple $\lceil \alpha, \beta, \gamma, \delta \rceil$ up to a common constant of proportionality. That is to designate

$$\lceil x_1, x_2, x_3, x_4 \rceil = \lceil \lambda x_1, \lambda x_2, \lambda x_3, \lambda x_4 \rceil \text{ and}$$

$$\lceil \alpha, \beta, \gamma, \delta \rceil = \lceil \lambda \alpha, \lambda \beta, \lambda \gamma, \lambda \delta \rceil.$$

Remark. If $x_4 \neq 0$, one may write

$$x = \frac{x_1}{x_4} = \frac{\lambda x_1}{\lambda x_4}, \quad y = \frac{x_2}{x_4} = \frac{\lambda x_2}{\lambda x_4}, \quad z = \frac{x_3}{x_4} = \frac{\lambda x_3}{\lambda x_4}.$$

---

*This is the empirically conceived three-dimensional space.

But what does the condition $x_4 \neq 0$ really mean? One can easily answer the question by taking $x_4 \to 0$. Then $x \to \infty$, $y \to \infty$ and $z \to \infty$. So the condition $x_4 \neq 0$ excludes the points at infinity. Since the configuration of the ground stations and the satellite positions is considered of finite extension, one can exclude the points at infinity without any consequences for the problem. When $x_4 \neq 0$ the convenient value $x_4 = 1$ is usually chosen, and the quadruples $\ulcorner x_1, x_2, x_3, 1 \urcorner$ are called affine coordinates. One then observes that $x = x_1$, $y = x_2$, $z = x_3$. Nevertheless, x_4 will be kept in the formulas for homogeneity and symmetry purposes, and it will be always assumed in the sequel that $x_4 \neq 0$. One might ask, whether it is necessary to exclude the points at infinity or not. The answer is not supposed to be given here, but anyway it is in the affirmative, for distance is not defined at infinity.

Going back after this digression, one observes that, points and planes have the same analytical representation. This is very important, because it allows for simultaneous analytical treatment of both the two sets E_3^ρ and E_3^π without having to distinguish between "point" and "plane" during the analysis. For a simultaneous analytical treatment of the sets E_3^ρ and E_3^π, a general set E is introduced, which designates the set of any Euclidean space to be defined below.

Having the non-empty set E, the notion of distance between the elements of the set naturally suggests itself. The non-empty set E equipped with a suitable concept of distance, (provided there exists a suitable one), is called Euclidean space. But what does one mean by distance, and what does this notion render? In the set E, a certain type of convergence is defined for the elements of the set. For example, in the sets E_3^ρ and E_3^π the following types of convergence are defined (see Figs. 5 and 6). Let $\ulcorner P_0, P_1, \ldots, P_i, \ldots \urcorner$ be a sequence of points in E_3^ρ. Then one says that this sequence converges to the point P, if the representations of the points $P_i \ulcorner x_i, y_i, z_i \urcorner$, $i = 0, 1, \ldots, i, \ldots$, converge to the representation of the point $P \ulcorner x, y, z \urcorner$, i.e.,

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$$\ulcorner x_0, x_1, \ldots, x_i, \ldots \urcorner \to x$$

$$\ulcorner y_0, y_1, \ldots, y_i, \ldots \urcorner \to y$$

$$\ulcorner z_0, z_1, \ldots, z_i, \ldots \urcorner \to z$$
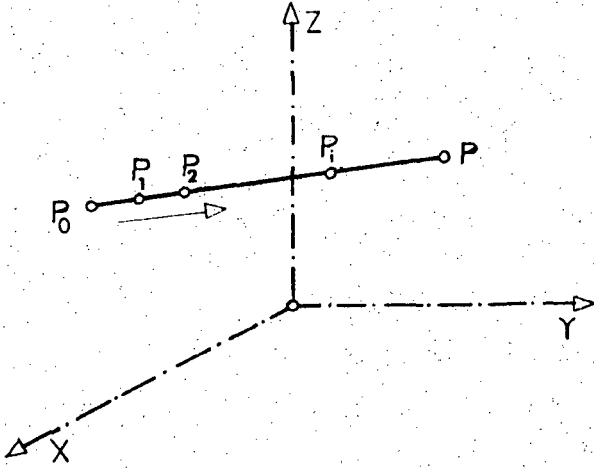
Fig. 5
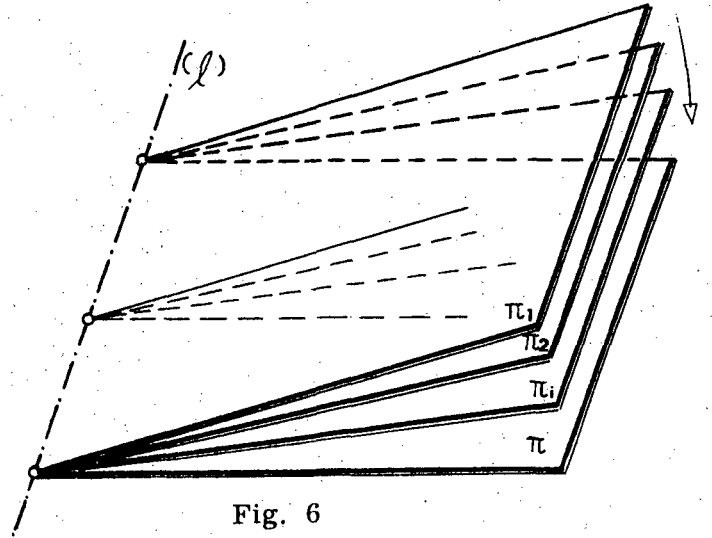
Fig. 6

Similarly, if $\pi_0, \pi_1, \ldots, \pi_i, \ldots$ is a sequence of planes in E_3^{π}, then one says that it converges to the plane π, if the representations of the planes $\pi_i \ulcorner \alpha_i, \beta_i, \gamma_i, \delta_i \urcorner$, $i = 0, 1,$ $\ldots, i, \ldots$, converge to the representation of the plane $\pi \ulcorner \alpha, \beta, \gamma, \delta \urcorner$, i.e.,

$$\ulcorner \alpha_0, \alpha_1, \ldots, \alpha_i, \ldots \urcorner \to \alpha$$

$$\ulcorner \beta_0, \beta_1, \ldots, \beta_i, \ldots \urcorner \to \beta$$

$$\ulcorner \gamma_0, \gamma_1, \ldots, \gamma_i, \ldots \urcorner \to \gamma$$

$$\ulcorner \delta_0, \delta_1, \ldots, \delta_i, \ldots \urcorner \to \delta.$$

Note: The above Figs. 5 and 6 illustrate two special types of convergence in E_3^p and E_3^{π} respectively, which are the only ones of interest in the problem in question. Fig. 5 illustrates the convergence of the sequence of the points

⌈P_0, P_1, ..., P_1, ...⌉ to the point P along the straight line of the points P_0 and P. In other words it illustrates the convergence of the sequence of the points which is generated by P_0, while it is approaching P along the straight line (P_0, P). Fig. 6 illustrates the convergence of the sequence of the planes ⌈π_0, π_1, ..., π_1, ...⌉, which is generated by π_0 while it is "approaching" the plane π, by turning around the line (ℓ) of the intersection of the planes π_0, and π. Now if there exists a function d of the coordinates of the elements of the set E which converges in an equivalent way to the convergence already defined in E, then d is called distance or metric. The set E along with the metric d, constitute a metric space designated by (E, d), which is the so called Euclidean space.

The question faced now is: does a suitable metric d exist in the set E such that convergence in the metric sense is equivalent to convergence in the above defined sense in the set E ? It is reminded that E stands for either of the sets E_3^p and E_3^{π} and since they have the same analytical representation their metrics (as functions of coordinates) will be of the same analytical expression interpreted in a dual way.

At this point a result of the general theory about the foundation of Euclidean geometry is recalled, in accordance with what was said at the beginning, about the discussion of this topic. Thus the general foundation theory of Euclidean geometry proves: The general expression of a Euclidean metric $d(P_0, P)$ in three-dimension Euclidean space is the non-negative real function

$$d(P_0, P) = \left[\sum_{i, k=1}^{3} g_{ik} \frac{\begin{vmatrix} x_i^0 & x_i \\ x_4^0 & x_4 \end{vmatrix} \cdot \begin{vmatrix} x_k^0 & x_k \\ x_4^0 & x_4 \end{vmatrix}}{(x_4^0 \cdot x_4)^2} \right]^{\frac{1}{2}} \qquad (18)$$

where [x_1^0, x_2^0, x_3^0, x_4^0] and [x_1, x_2, x_3, x_4] are the analytic representation of the elements P_0 and P. One may think of P_0 and P either as points, or planes, or any other equally justifiable configuration, with the same analytical representation. In non-homogeneous coordinates, expression (18) has the form

$$d(P_0, P) = \left[\sum_{\substack{i, k=1}}^{3} g_{ik} (x_i^0 - x_i)(x_k^0 - x_k) \right]^{\frac{1}{2}}, \tag{18}'$$

which is obtained by choosing for the non-vanishing x_4 the value of one.
Although (18)' is simpler than (18), the latter is more useful for the problem in
question. Before specifying the expression (18) for the cases of interest here,
(points and planes), a few general remarks about the Euclidean metrics are in
order:

(a) The square of the Euclidean metric $d(P_0, P)$ is a positive definite quadratic
form. Expression (18) may be written in matrix notation as follows.

$$d^2(P_0, P) = \sum_{i, k=1}^{3} g_{ik} \frac{\begin{vmatrix} x_i^0 & x_i \\ x_4^0 & x_4 \end{vmatrix} \cdot \begin{vmatrix} x_k^0 & x_k \\ x_4^0 & x_4 \end{vmatrix}}{(x_4^0 \cdot x_4)^2} = X_{P_0P}^{\tau} G \, X_{P_0P} \tag{19}$$

where

$$G = g_{ik} = \begin{bmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{bmatrix}, \qquad G = G^{\tau} \tag{20}$$

and

$$X_{P_0P}^{\tau} = \left[\frac{\begin{vmatrix} x_1^0 & x_1 \\ x_4^0 & x_4 \end{vmatrix}}{x_4^0 \cdot x_4} \quad \frac{\begin{vmatrix} x_2^0 & x_2 \\ x_4^0 & x_4 \end{vmatrix}}{x_4^0 \cdot x_4} \quad \frac{\begin{vmatrix} x_3^0 & x_3 \\ x_4^0 & x_4 \end{vmatrix}}{x_4^0 \cdot x_4} \right], \qquad x_4 \neq 0. \tag{21}$$

In the case of non-homogeneous coordinates, (21) has the form

$$X_{P_0P}^{\tau} = \left[(x_1^0 - x_1), (x_2^0 - x_2), (x_3^0 - x_3) \right]$$
$$= \left[(x^0 - x), (y^0 - y), (z^0 - z) \right]. \tag{21}'$$

One says that the quadratic form $d^2(P_0, P) = X^{\tau}G\,X$ is positive definite, when
$X^{\tau}G\,X > 0$ for every $X \neq 0$. The same can be expressed in terms of the
matrix G. Thus $X^{\tau}G\,X$ is positive definite when G is positive definite, meaning that
it has positive eigenvalues. Since G is a real symmetric matrix, if λ_1, λ_2, λ_3

are its eigenvalues with p_1, p_2, p_3 the corresponding normalized* eigenvectors, then

$$P^T G P = \Lambda = \text{diag}(\lambda_1, \lambda_2, \lambda_3); \tag{22}$$

where $P = [p_1, p_2, p_3]$ is the matrix whose columns are the normalized eigenvectors of G, and Λ is a diagonal matrix whose diagonal elements are the eigenvalues of G. It is a well-known result that P is an orthogonal matrix, and $P^T P = PP^T = I$. Also $P^T = P^{-1}$.

(b) In the previous subsection it was said that the motions of Euclidean geometry preserve the distances and angles. The converse is also true. Having now both of these measures (distances and angles) expressed analytically under one expression $d(P_0, P)$, one can state that the motions of Euclidean geometry are those which preserve the value of $d(P_0, P)$ for any pair of elements (P_0, P), provided none of P_0 and P is at infinity. Let S be a matrix of a Euclidean motion. It is interesting to find out what kind of matrix S is. Consider the "distance" $d(P_0, P)$. Under the motion S, $d(P_0, P)$ is by definition an invariant quantity. If Y_{P_0P} is the new representation vector, into which X_{P_0P} is transformed under S, one has

$$Y_{P_0P} = S X_{P_0P}. \tag{23}$$

Then

$$d(P_0, P) = X_{P_0P}^T G X_{P_0P} = (S^{-1} Y_{P_0P})^T G (S^{-1} Y_{P_0P})$$
$$= Y_{P_0P}^T [(S^{-1})^T G (S^{-1})] Y_{P_0P}$$
$$= Y_{P_0P}^T M Y_{P_0P}$$

where

$$M = (S^{-1})^T G (S^{-1}).$$

The invariant character of $d(P_0, P)$ entails

$$M = G$$

or

$$(S^{-1})^T G (S^{-1}) = G$$

or

$$G = S^T G S. \tag{24}$$

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*If p_1^*, p_2^*, p_3^* are the eigenvectors of G, then the normalized ones are

$$p_1 = \frac{p_1^*}{||p_1^*||}, \quad p_2 = \frac{p_2^*}{||p_2^*||}, \quad p_3 = \frac{p_3^*}{||p_3^*||}.$$

(c) One point of importance here is the following. In the case of a general metric space, analysis defines a metric d on a set E* as a real function of ordered pairs of elements of E which satisfies the following three conditions:

(i) $d(e_1, e_2) \geq 0$, and $d(e_1, e_2) = 0$ if $e_1 = e_2$;

(ii) $d(e_1, e_2) = d(e_2, e_1)$, (symmetry);

(iii) $d(e_1, e_2) \leq d(e_1, e_3) + d(e_3, e_2)$, (the triangle inequality).

The above given Euclidean metric obviously satisfies the conditions of this definition. Although on one and the same set E, one may define many metrics in the above sense, each one renders different metrization of E. As it was said, the introduction of a metric follows the definition of a certain type of convergence for the sequences of the elements of the set. It is the type of convergence defined on the set which constitutes the criterion for the suitability of a metric.

The Euclidean metric given by (18) is one of the so-called <u>projective metrics.</u> The metrics of all the metric geometries derived from projective geometry are characterized as projective metrics. Thus, the use in some problem of a metric whose form is not that of (18) means that the problem is not treated within the Euclidean geometry goemetry. For example, the metric

$$d_1(P_0, P) = \max_{i=1, 2, 3} \frac{\left| \begin{matrix} x_i^0 & x_i \\ x_4^0 & x_4 \end{matrix} \right|}{(x_4^0 x_4)^2}, \quad (x_4^0, x_4 \neq 0),$$

or for $x_4^0 = x_4 = 1$

$$d_1(P_0, P) = \max_{i=1, 2, 3} \left| x_i^0 - x_i \right| \tag{25}$$

does satisfy the conditions of the above definition of a general metric, but it is not a projective metric. The same with the metric

$$d_2(P_0, P) = \sum_{i=1}^{3} \left| \begin{matrix} x_i^0 & x_i \\ x_4^0 & x_4 \end{matrix} \right| \frac{1}{x_4^0 x_4}, \quad (x_4^0, x_4 \neq 0)$$

*Here E stands for an abstract set.

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or for $x_4^0 = x_4 = 1$

$$d_2(P_0, P) = \sum_{i=1}^{3} |x_i^0 - x_i| .$$

(26)

(d) In this note the corresponding general expression for a Euclidean metric in two dimensions will be given. This can be done immediately by omitting the coordinate x_4 and considering the remaining coordinates x_1, x_2, x_3 as homogeneous in two dimensions, i.e.,

$$x = \frac{x_1}{x_3}, \qquad y = \frac{x_2}{x_3}, \qquad x_3 \neq 0.$$

Then

$$d(P_0, P) = \left[\sum_{i,k=1}^{2} g_{i,k} \frac{\begin{vmatrix} x_i^0 & x_i \\ x_3^0 & x_3 \end{vmatrix} \begin{vmatrix} x_k^0 & x_k \\ x_3^0 & x_3 \end{vmatrix}}{(x_3^0 \cdot x_3)^2} \right]^{\frac{1}{2}} ,$$

(27)

or in matrix notation

$$d^2(P_0, P) = X_{P_0 P}^T \, G_2 \, X_{P_0 P},$$

(27)′

where

$$G_2 = (g_{ik}) = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} , \qquad G_2 = G_2^T,$$

and

$$X_{P_0 P} = \left[\frac{\begin{vmatrix} x_1^0 & x_1 \\ x_3^0 & x_3 \end{vmatrix}}{x_3^0 \cdot x_3} , \frac{\begin{vmatrix} x_2^0 & x_2 \\ x_3^0 & x_3 \end{vmatrix}}{x_3^0 \cdot x_3} \right] , \qquad x_3 \neq 0.$$

(28)

In non-homogeneous coordinates, (27) and (28) have the forms

$$d(P_0, P) = \left[\sum_{i,k=1}^{2} g_{ik} (x_i^0 - x_i)(x_k^0 - x_k) \right]^{\frac{1}{2}}$$

(27)″

and

$$X_{P_0 P} = [(x_1^0 - x_1), (x_2^0 - x_2)]$$
$$= [(x^0 - x), (y^0 - y)]$$

(28)′

2.25 The Usual Euclidean Metric and Its Dual

The general expression (18) of a Euclidean metric is recalled here

$$d(P_0, P) = \left[\sum_{i, k=1}^{3} g_{ik} \frac{\begin{vmatrix} x_i^0 & x_i \\ x_i^4 & x_4 \end{vmatrix} \begin{vmatrix} x_k^0 & x_k \\ x_4^0 & x_4 \end{vmatrix}}{(x_4^0 \cdot x_4)^2} \right]^{\frac{1}{2}}, \qquad (x_4^0, x_4 \neq 0) \tag{29}$$

where $[x_1^0, x_2^0, x_3^0, x_4^0]$ and $[x_1, x_2, x_3, x_4]$ are the analytic representations of the elements P_0 and P. At this subsection the elements P_0 and P will be interpreted either as points or planes. The matrix formulation of the above expression is recalled also from (19), (20), and (21).

$$d^2(P_0, P) = X_{P_0 P}^T \, G \, X_{P_0 P}, \tag{29}'$$

$$G = \begin{bmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{bmatrix}, \quad G = G^T$$

$$X_{P_0 P}^T = \left[\frac{\begin{vmatrix} x_1^0 & x_1 \\ x_4^0 & x_4 \end{vmatrix}}{x_4^0 \cdot x_4}, \frac{\begin{vmatrix} x_2^0 & x_2 \\ x_4^0 & x_4 \end{vmatrix}}{x_4^0 \cdot x_4}, \frac{\begin{vmatrix} x_3^0 & x_3 \\ x_4^0 & x_4 \end{vmatrix}}{x_4^0 \cdot x_4} \right], \qquad (x_4^0, x_4 \neq 0) \tag{30}$$

While discussing the principal of duality in the three-dimension projective space, it was shown that "point" and "plane" are dual notions. It is true that the duality principle does not hold in Euclidean geometry in general. Thus whenever duality principle is drafted in Euclidean geometry, its validity must be demonstrated sufficiently. That is why the preceding analysis has been placed before this point.

If in the general expression of a Euclidean metric one takes the elements P_0 and P as points, then $d(P_0, P)$ renders a means of "distance"* between the points P_0 and P. If one takes P_0 and P as planes, then $d(P_0, P)$ renders a means of "distance" between the two planes. In the following, the first case will be considered first, for it is simpler and more visual than the second, in order to find out how the usual Euclidean metric results from the general expression (29). Then the second

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*The quotation marks are justified by the fact that the word "distance" stands for the usual Euclidean metric, which, as it is proved in the sequel, is a special case of the above.

case will be deployed on the first, using the duality principle as a guide, to find out the geometrical meaning of the dual metric to the usual Euclidean metric.

Case 1. Point-Euclidean space (E_3^p, d).

In this case the elements of the space are the points* of E_3^p, which the human mind seems to visualize better than any other configuration. The usual Euclidean metric, i. e., the usual expression for the distance between two points, will be connected with the general expression of a Euclidean metric.

Let P_0 and P be two points with Cartesian coordinates $[x_1^0, x_2^0, x_3^0]$ and $[x_1, x_2, x_3]$ respectively. Then the usual Euclidean metric has the form

$$d(P_0, P) = [(x_1^0 - x_1)^2 + (x_2^0 - x_2)^2 + (x_3^0 - x_3)^2]^{\frac{1}{2}}. \tag{31}$$

Introducing homogeneous coordinates, the last expression may be written

$$d(P_0, P) = \left[\frac{\begin{vmatrix} x_1^0 & x_1 \\ x_4^0 & x_4 \end{vmatrix}^2}{(x_4^0 x_4)^2} + \frac{\begin{vmatrix} x_2^0 & x_2 \\ x_4^0 & x_4 \end{vmatrix}^2}{(x_4^0 x_4)^2} + \frac{\begin{vmatrix} x_3^0 & x_3 \\ x_4^0 & x_4 \end{vmatrix}^2}{(x_4^0 x_4)^2} \right]^{\frac{1}{2}}, \tag{31'}$$

or

$$d(P_0, P) = \left[X_{P_0 P}^T \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} X_{P_0 P} \right]^{\frac{1}{2}} = \left[X_{P_0 P}^T X_{P_0 P} \right]^{\frac{1}{2}}, \tag{32}$$

where

$$X_{P_0 P} = \left[\frac{\begin{vmatrix} x_1^0 & x_1 \\ x_4^0 & x_4 \end{vmatrix}}{x_4^0 x_4}, \frac{\begin{vmatrix} x_2^0 & x_2 \\ x_4^0 & x_4 \end{vmatrix}}{x_4^0 x_4}, \frac{\begin{vmatrix} x_3^0 & x_3 \\ x_4^0 & x_4 \end{vmatrix}}{x_4^0 x_4} \right].$$

Thus the usual Euclidean metric results from the general expression of a Euclidean metric for $G = I_3$, i. e., when G is the identity matrix. Recalling the relation (24) of the remark (b) which followed the general expression (18), we observe that the matrix S of a motion in Euclidean geometry satisfies the relation

$$S\,S^T = S^T S = I_3 \tag{33}$$

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* It is recalled that when an element is considered as the generating element of the space, it is supposed to be undefined for the geometry to be constructed with that.

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That means that S is orthogonal. Thus the motions in Euclidean geometry with the usual Euclidean metric are analytically represented by orthogonal matrices. Analogously, in the two-dimensional Euclidean space, the usual Euclidean metric results from the general expression (27)´ for

$$G_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2.$$

Case 2.

In this case the elements of the space are the planes of E_3^π. Although they have the same analytical representation with points, it is much more difficult to visualize configurations of planes than of points. Therefore, duality contributes substantially to problems with configurations of planes like the present one. The dual metric of the usual Euclidean metric will be formed and its geometrical meaning will be given.

Let π_0 and π be two planes with analytical representations $\lceil \alpha_0, \beta_0, \gamma_0, \delta_0 \rfloor$ and $\lceil \alpha, \beta, \gamma, \delta \rfloor$ respectively. Their equations are

$$\alpha_0 x_1 + \beta_0 x_2 + \gamma_0 x_3 + \delta_0 x_4 = 0,$$
and   (33)
$$\alpha x_1 + \beta x_2 + \gamma x_3 + \delta x_4 = 0.$$

Since the points at infinity are not considered in the problem (i. e., $x_4 \ne 0$), one could write these equations as follows:

$$\alpha_0 x_1 + \beta_0 x_2 + \gamma_0 x_3 + \delta_0 = 0,$$
and   (33)´
$$\alpha x_1 + \beta x_2 + \gamma x_3 + \delta = 0.$$

They resulted from the previous ones for $x_4 = 1$. By taking now the elements P_0 and P in the general expression (29)´ of a Euclidean metric, as the planes π_0 and π one has

$$d(\pi_0, \pi) = [(\Omega_{\pi_0 \pi})^\tau G(\Omega_{\pi_0 \pi})]^{\frac{1}{2}}$$ (34)

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where

$$(\Omega_{\pi_0 \pi})^\top = \begin{bmatrix} \begin{vmatrix} \alpha_0 & \alpha \\ \delta_0 & \delta \end{vmatrix}, & \begin{vmatrix} \beta_0 & \beta \\ \delta_0 & \delta \end{vmatrix}, & \begin{vmatrix} \gamma_0 & \gamma \\ \delta_0 & \delta \end{vmatrix} \\ \delta_0 \delta & \delta_0 \delta & \delta_0 \delta \end{bmatrix}, \quad (\delta_0, \delta \neq 0) \tag{35}$$

and G remains exactly the same, for it is independent of the given interpretation to the representation vector $X_{P_0 P}$. Now for

$$G = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3,$$

relation (34) yields

$$d(\pi_0, \pi) = [(\Omega_{\pi_0, \pi})^\top (\Omega_{\pi_0, \pi})]^{\frac{1}{2}}. \tag{36}$$

This metric is a Euclidean metric, and, consequently, a projective one. It expresses the "distance" between π_0 and π in the same way the metric $d(P_0, P)$ in (32) expresses the distance between the points P_0 and P. But what is the geometrical meaning of saying "the plane π"? Any comment on this question is postponed until later. Now the geometrical meaning of $d(\pi_0, \pi)$ is in order. While trying to interpret $d(\pi_0, \pi)$ geometrically, the above question will manifest itself in a natural way.

Let

$$\rho_0 = [\alpha_0^2 + \beta_0^2 + \gamma_0^2]^{\frac{1}{2}},$$

and

$$\rho = [\alpha^2 + \beta^2 + \gamma^2]^{\frac{1}{2}}.$$

Then the expression (35) for $(\Omega_{\pi_0, \pi})^\top$ may be written as follows.

$$(\Omega_{\pi_0, \pi})^\top = \begin{bmatrix} \begin{vmatrix} \frac{\alpha_0}{\rho_0} & \frac{\alpha}{\rho} \\ \frac{\delta_0}{\rho_0} & \frac{\delta}{\rho} \end{vmatrix}, & \begin{vmatrix} \frac{\beta_0}{\rho_0} & \frac{\beta}{\rho} \\ \frac{\delta_0}{\rho_0} & \frac{\delta}{\rho} \end{vmatrix}, & \begin{vmatrix} \frac{\gamma_0}{\rho_0} & \frac{\gamma}{\rho} \\ \frac{\delta_0}{\rho_0} & \frac{\delta}{\rho} \end{vmatrix} \\ \frac{\delta_0}{\rho_0} \cdot \frac{\delta}{\rho} & \frac{\delta_0}{\rho_0} \cdot \frac{\delta}{\rho} & \frac{\delta_0}{\rho_0} \cdot \frac{\delta}{\rho} \end{bmatrix} \tag{37}$$

where both terms of each fraction have been multiplied by $1/(\rho_0\rho)$. Since $d(\pi_0,\pi)$ is an invariant under Euclidean motions(see section 2.23), a parallel displacement is applied on the configuration of the two planes π_0 and π such that

$$\frac{\delta_0}{\rho_0} = \frac{\delta}{\rho} = 1.$$

This transformation, though applicable, is not very clear from the present point of view. Therefore, another proof, purely analytical, will be given. The proof to follow, though rigorous, is in informal language in order to be put within the background limits already set previously.

It was shown in (section 2.21) that each plane (π) determines the quadruple $\lceil\alpha, \beta, \gamma, \delta\rceil$ up to a common constant of proportionality. That means that the planes $\lceil\alpha, \beta, \gamma, \delta\rceil$ and $\lceil\lambda\alpha, \lambda\beta, \lambda\gamma, \lambda\delta\rceil$, $\lambda \neq 0$, which are really parallel to each other in the Euclidean space, are considered one and the same plane in the projective space, or as one says in formal language, they belong to the same class*. Therefore, each one of the planes π_0 and π determines a class in the projective space, namely, they determine the classes $\lceil\lambda\alpha_0, \lambda\beta_0, \lambda\gamma_0, \lambda\delta_0\rceil$ and $\lceil\mu\alpha, \mu\beta, \mu\gamma, \mu\delta\rceil$ respectively. Since the general Euclidean metric, where the metric $d(\pi_0,\pi)$ comes from, is a projective metric(see section 2.24), $d(\pi_0,\pi)$ is a suitable metric for expressing the convergence of a sequence of "points" in the projective space, i.e., $d(\pi_0,\pi)$ can be used to express the convergence of a sequence of classes. Consequently, $d(\pi_0,\pi)$ expresses the "distance" between the classes $\lceil\lambda\alpha_0, \lambda\beta_0, \lambda\gamma_0, \lambda\delta_0\rceil$ and $\lceil\mu\alpha, \mu\beta, \mu\gamma, \mu\delta\rceil$, independently** from the values of λ and μ. Thus, that "distance" is the same for any pair of planes from the two classes. Just for convenience one may choose the pair of planes for which $\delta_0/\rho_0 = \delta/\rho = 1$. Note that the planes $\delta_0 = 0$ and $\delta = 0$ are excluded from the respective classes.

The above informal proof, indicates how analysis treats some intuitively appealing facts in Euclidean geometry. Another important thing here is the projective character of the Euclidean metric.

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*The "points" of the projective space are the classes $\lceil\lambda\alpha, \lambda\beta, \lambda\gamma, \lambda\delta\rceil$, $(\lambda \neq 0)$. The class is completely determined if only one of its members is known. The others are obtained by varying λ.

**λ and μ keep their signs while they vary independently.

By putting $\delta_0/\rho_0 = \delta/\rho = 1$ in the expression (37) of $(\Omega_{\pi_0, \pi})$, the following is obtained.

$$(\Omega_{\pi_0, \pi})^{\mathsf{T}} = \left[\left(\frac{\alpha_0}{\rho_0} - \frac{\alpha}{\rho} \right), \left(\frac{\beta_0}{\rho_0} - \frac{\beta}{\rho} \right), \left(\frac{\gamma_0}{\rho_0} - \frac{\gamma}{\rho} \right) \right] \tag{37}$$

Then the expression (36) of $d(\pi_0, \pi)$ becomes

$$d^2(\pi_0, \pi) = \left(\frac{\alpha_0}{\rho_0} - \frac{\alpha}{\rho} \right)^2 + \left(\frac{\beta_0}{\rho_0} - \frac{\beta}{\rho} \right)^2 + \left(\frac{\gamma_0}{\rho_0} - \frac{\gamma}{\rho} \right)^2 \tag{38}$$
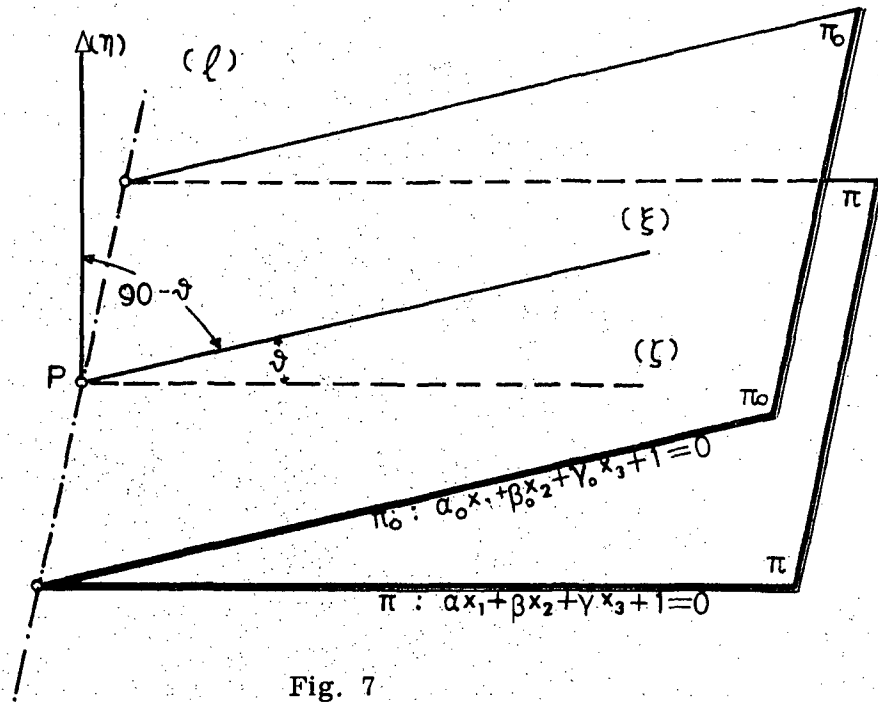

Fig. 7

Consider now the two planes

$$\pi_0 : \quad \alpha_0 x_1 + \beta_0 x_2 + \gamma_0 x_3 + \delta_0 = 0, \text{ and} \tag{39}$$

$$\pi : \quad \alpha x_1 + \beta_1 x_2 + \gamma_1 x_3 + \delta = 0, \tag{40}$$

as well as the lines

    ℓ intersection of π_0 and π

    ξ perpendicular to ℓ in π_0

ζ perpendicular to l in π, and

η perpendicular to π.

Without loss of generality, consider these lines being concurrent at the point P of l. Let the following notation be introduced:

X_1^P, X_2^P, X_3^P coordinates of P,

a_l, b_l, c_l direction-cosines of l,

a_ξ, b_ξ, c_ξ direction-cosines of ξ,

a_ζ, b_ζ, c_ζ direction-cosines of ζ, and

a_η, b_η, c_η direction-cosines of η.

Then the equations of π_0 and π, as the planes which pass through P and are parallel to the pairs of lines $[\xi, l]$ and $[\zeta, l]$ respectively, are

$$\pi_0: \quad \begin{vmatrix} x_1 - X_1^P & x_2 - X_2^P & x_3 - X_3^P \\ a_\xi & b_\xi & c_\xi \\ a_l & b_l & c_l \end{vmatrix} = 0, \tag{41}$$

and

$$\pi: \quad \begin{vmatrix} x_1 - X_1^P & x_2 - X_2^P & x_3 - X_3^P \\ a_\zeta & b_\zeta & c_\zeta \\ a_l & b_l & c_l \end{vmatrix} = 0. \tag{42}$$

Comparing (39) with (41) and (40) with (42), the following is obtained:

$$\alpha_0 = s_0 \begin{vmatrix} b_\xi & c_\xi \\ b_l & c_l \end{vmatrix}, \quad \alpha = s \begin{vmatrix} b_\zeta & c_\zeta \\ b_l & c_l \end{vmatrix}, \tag{43}$$

$$\beta_0 = s_0 \begin{vmatrix} c_\xi & a_\xi \\ c_l & a_l \end{vmatrix}, \quad \beta = s \begin{vmatrix} c_\zeta & a_\zeta \\ c_l & a_l \end{vmatrix},$$

$$\gamma_0 = s_0 \begin{vmatrix} a_\xi & b_\xi \\ a_l & b_l \end{vmatrix}, \quad \gamma = s \begin{vmatrix} a_\zeta & b_\zeta \\ a_l & b_l \end{vmatrix},$$

$$\delta_0 = -s_0 \begin{vmatrix} X_1^P & X_2^P & X_3^P \\ a_\xi & b_\xi & c_\xi \\ a_\ell & b_\ell & c_\ell \end{vmatrix}, \qquad \delta = -s \begin{vmatrix} X_1^Q & X_2^Q & X_3^Q \\ a_\zeta & b_\zeta & c_\zeta \\ a_\ell & b_\ell & c_\ell \end{vmatrix},$$

where s_0 and s are constants of proportionality. Notice the presence of the point (X_1^Q, X_2^Q, X_3^Q); Q is on ℓ, generally different from P. The following theorem is useful at this point.

Theorem. The square of the sine of the angle between two directed lines is equal to the sum of the squares of the two-rowed minors which can be formed from the matrix constituted by the two sets of direction cosines of the lines.

If μ and ν are two lines with direction cosines $[a_\mu, b_\mu, c_\mu]$ and $[a_\nu, b_\nu, c_\nu]$ respectively, the above theorem proves that

$$\sin^2(\mu, \nu) = \sin^2\theta = \begin{vmatrix} b_\mu & c_\mu \\ b_\nu & c_\nu \end{vmatrix}^2 + \begin{vmatrix} c_\mu & a_\mu \\ c_\nu & a_\nu \end{vmatrix}^2 + \begin{vmatrix} a_\mu & b_\mu \\ a_\nu & b_\nu \end{vmatrix}^2. \qquad (44)$$

Proof. Consider the identity

$$(a_1^2 + b_1^2 + c_1^2)(a_2^2 + b_2^2 + c_2^2) - (a_1 a_2 + b_1 b_2 + c_1 c_2)^2 =$$

$$= \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}^2 + \begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}^2 + \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}^2. \qquad (45)$$

Applying this to the direction cosines of the lines μ and ν, the following is obtained.

$$1 \cdot 1 - (a_\mu a_\nu + b_\mu b_\nu + c_\mu c_\nu)^2 = \begin{vmatrix} b_\mu & c_\mu \\ b_\nu & c_\nu \end{vmatrix}^2 + \begin{vmatrix} c_\mu & a_\mu \\ c_\nu & a_\nu \end{vmatrix}^2 + \begin{vmatrix} a_1 & b_1 \\ a_\nu & b_\nu \end{vmatrix}^2$$

But

$$a_\mu a_\nu + b_\mu b_\nu + c_\mu c_\nu = \cos(\mu, \nu) = \cos\theta$$

Then the left-hand side of the last relation is $\sin^2\theta$, and thus the theorem is proved.

By virtue of this theorem, relations (43) yield

$$\alpha_0^2 + \beta_0^2 + \gamma_0^2 = s_0^2 \sin^2(\xi, \ell) = s_0^2 \sin^2 90° = s_0^2,$$

and

$$\alpha^2 + \beta^2 + \gamma^2 = s^2 \sin^2(\zeta, \ell) = s^2 \sin^2 90° = s^2.$$

Thus

$$\rho_0 = s_0 \quad \text{and} \quad \rho = s.$$

Relations (43) yield the normalized coordinates of the planes π_0 and π in terms of the direction cosines of three characteristic lines (l), (ξ), and (ζ). This representation is very useful for it offers the means to transfer from plane to line coordinates and vice-versa.

$$\frac{\alpha_0}{\rho_0} = \begin{vmatrix} b_\xi & c_\xi \\ b_l & c_l \end{vmatrix} \; ; \qquad \frac{\alpha}{\rho} = \begin{vmatrix} b_\zeta & c_\zeta \\ b_l & c_l \end{vmatrix} \; ,$$

$$\frac{\beta_0}{\rho_0} = \begin{vmatrix} c_\xi & a_\xi \\ c_l & a_l \end{vmatrix} \; ; \qquad \frac{\beta}{\rho} = \begin{vmatrix} c_\zeta & a_\zeta \\ c_l & a_l \end{vmatrix} \tag{46}$$

$$\frac{\gamma_0}{\rho_0} = \begin{vmatrix} a_\xi & b_\xi \\ a_l & b_l \end{vmatrix} \; ; \qquad \frac{\gamma}{\rho} = \begin{vmatrix} a_\zeta & b_\zeta \\ a_l & b_l \end{vmatrix}$$

$$\frac{\delta_0}{\rho_0} = - \begin{vmatrix} X_1^P & X_2^P & X_3^P \\ a_\xi & b_\xi & c_\xi \\ a_l & b_l & c_l \end{vmatrix} \; ; \qquad \frac{\delta}{\rho} = - \begin{vmatrix} X_1^Q & X_2^Q & X_3^Q \\ a_\zeta & b_\zeta & c_\zeta \\ a_l & b_l & c_l \end{vmatrix} \; .$$

One may introduce the last relations into (38) to express $d^2(\pi_0, \pi)$ in terms of the directional cosines of the lines (l), (ξ) and (ζ). However, the discussion continues at this point in plane coordinates.

The expression (38) of $d^2(\pi_0, \pi)$ may be written

$$d^2(\pi_0, \pi) = \frac{\alpha_0^2 + \beta_0^2 + \gamma_0^2}{\rho_0^2} + \frac{\alpha^2 + \beta^2 + \gamma^2}{\rho^2} - 2\frac{\alpha_0\alpha + \beta_0\beta + \gamma_0\gamma}{\rho_0\rho}$$

$$= 1 + 1 - 2\frac{\alpha_0\alpha + \beta_0\beta + \gamma_0\gamma}{\rho_0\rho}$$

or

$$d^2(\pi_0, \pi) = 2\left(1 - \frac{\alpha_0\alpha + \beta_0\beta + \gamma_0\gamma}{\rho_0\rho}\right). \tag{47}$$

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As it is well-known, the vectors $[\alpha_0, \beta_0, \gamma_0]$ and $[\alpha, \beta, \gamma]$ are perpendicular to the planes π_0 and π, respectively. But each plane has two faces, and if $[\alpha, \beta, \gamma]$ is pointing outward on one face of π, then the vector $[-\alpha, -\beta, -\gamma]$ is pointing outward on the other face of π.
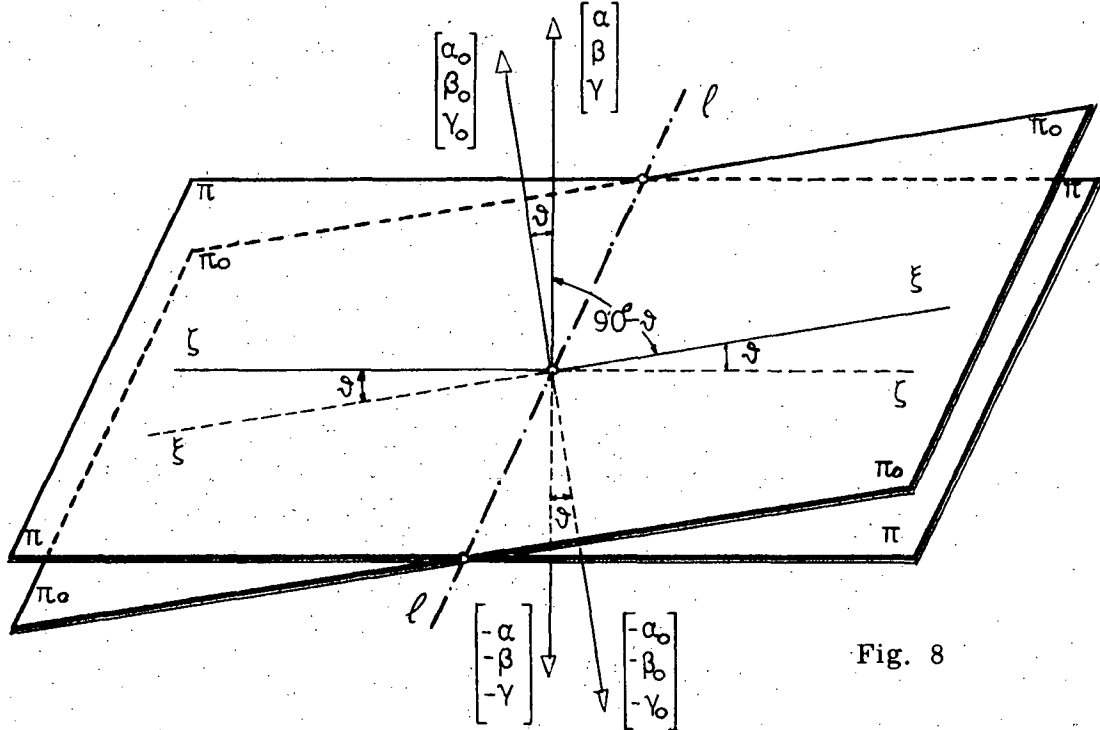
Fig. 8

Let each face of a plane be represented by the corresponding perpendicular vector to the plane. The the following notation is adopted.

$$\pi_0 \equiv \frac{1}{\rho_0} [\alpha_0, \beta_0, \gamma_0] \ , \qquad -\pi_0 \equiv \frac{1}{\rho_0} [-\alpha_0, -\beta_0, -\gamma_0]$$

$$\pi \equiv \frac{1}{\rho_0} [\alpha, \beta, \gamma] \ , \qquad -\pi \equiv \frac{1}{\rho} [-\alpha, -\beta, -\gamma]$$

Therefore, when one talks about "distance" of two planes, he actually means "distance" between two of the four faces. When the equation of a plane is written down, the face of the plane to be considered is defined. Consequently, that equation is for one of the faces of the plane. This remark sounds probably trivial, but it is of importance at this point to the advancement of the present analysis.

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Let θ be the angle* between the vectors $[\alpha_0, \beta_0, \gamma_0]$ and $[\alpha, \beta, \gamma]$, that is the angle between the faces corresponding to these vectors. Then one has

$$\frac{\alpha_0}{\rho_0} \cdot \frac{\alpha}{\rho} + \frac{\beta_0}{\rho_0} \cdot \frac{\beta}{\rho} + \frac{\gamma_0}{\rho_0} \cdot \frac{\gamma}{\rho} = \cos\theta ,$$

and the relation (47) is written

$$d^2(\pi_0, \pi) = 2(1 - \cos\theta) = 4\sin^2\frac{\theta}{2} . \tag{48}$$

This is the "distance" between the faces $[\alpha_0, \beta_0, \gamma_0]$ and $[\alpha, \beta, \gamma]$. Consider now the "distance" between the faces $[\alpha, \beta, \gamma]$ and $[-\alpha_0, -\beta_0, -\gamma_0]$. Equation (47) gives

$$d^2(-\pi_0, \pi) = 2\left(1 - \frac{(-\alpha_0)\alpha + (-\beta_0)\beta + (-\gamma_0)\gamma}{\rho_0 \rho}\right) ,$$

or

$$d^2(-\pi_0, \pi) = 2(1 + \cos\theta) = 4\cos^2\frac{\theta}{2} . \tag{49}$$

Thus, one observes that given a face $\pi = [\alpha, \beta, \gamma]$ one may define with the two faces of another plane π_0 two different "distances." As a special case, one can consider the "distance" between the faces of a plane π_0; that is

$$d^2(\pi_0, -\pi_0) = 2\left[1 - \frac{\alpha_0(-\alpha_0) + \beta_0(-\beta_0) + \gamma(-\gamma_0)}{\rho_0 \rho_0}\right] = 2(1 + 1) = 4 ,$$

i.e.,

$$d^2(\pi_0, -\pi_0) = 4 . \tag{50}$$

Recapitulating the above formulas about "distances" between the faces of two planes one has

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*One may find the notion "angle" out of place in this context. However, this discussion aims at revealing the geometrical meaning of the "distance" between two planes, and this mixture is unavoidable.

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$$d(\pi_0, \pi) = d(-\pi_0, -\pi) = \sqrt{2(1 - \cos\theta)} = 2\left|\sin\frac{\theta}{2}\right|$$

$$d(\pi_0, \pi) = d(\pi_0, -\pi) = \sqrt{2(1 - \cos\theta)} = 2\left|\cos\frac{\theta}{2}\right|$$

(51)

$$d(\pi_0, \pi_0) = d(\pi, \pi) = 0$$

$$d(\pi_0, -\pi_0) = d(\pi, -\pi) = 2$$

In order to comprehend better the discussion following this point, a geometrical (or trigonometrical) interpretation of the formulas (51) is introduced here. Consider the cylinder with axis the line of intersection (ℓ) of the planes π_0 and π and cross-section the unit circle. The cylinder is considered of infinite length (Fig. 9).
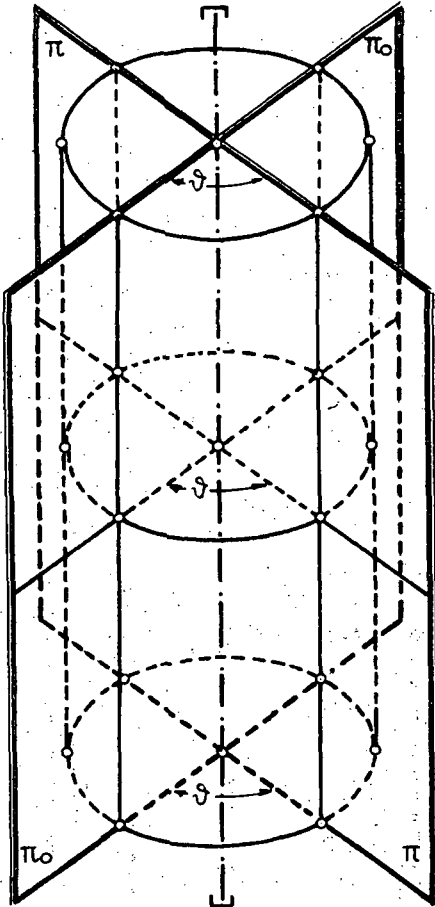

Fig. 9
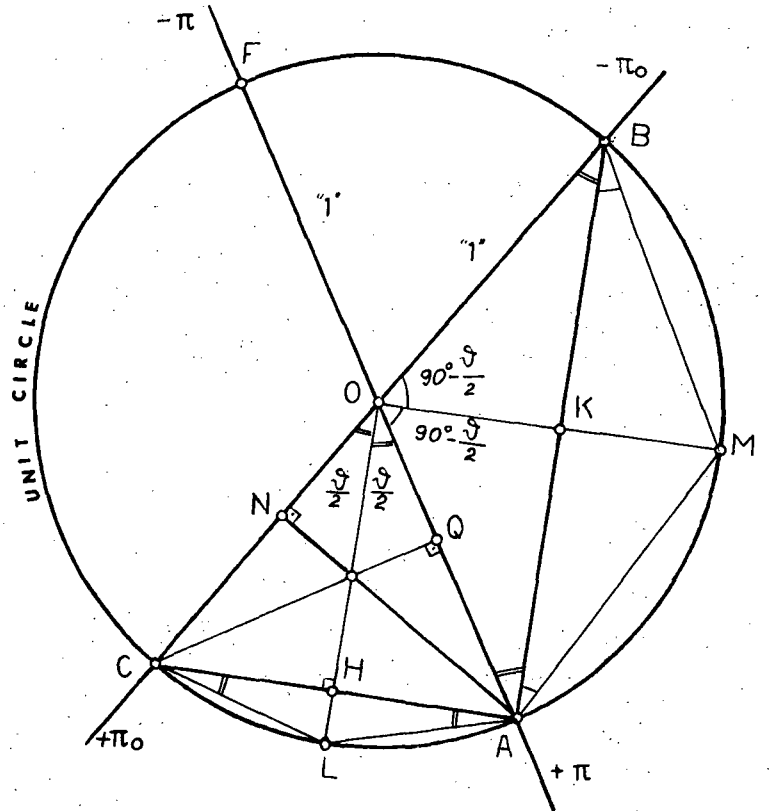
Fig. 10

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Take a cross-section of the cylinder (Fig. 10). This is a
unit circle with the traces (AF) and (BC) of the planes π and π_0, respectively. Using elementary trigonometry it is obtained that:

$$d(\pi_0, \pi) = d(-\pi_0, -\pi) = 2 \left| \sin\frac{\theta}{2} \right| = 4 \cdot \text{area (OCL)} = (AC)$$

$$\tag{52}$$

$$d(-\pi_0, \pi) = d(\pi_0, -\pi) = 2 \left| \cos\frac{\theta}{2} \right| = 4 \cdot \text{area (OAM)} = (AB)$$

Now, one has in his hands a very simple model in order to study the "distances" between the faces of two planes. The four faces $+\pi$, $-\pi$, $+\pi_0$, $-\pi_0$ are modeled by the points A, F, C, B, respectively on the circumference of the unit circle, and the "distance" between any two of the faces is modeled by the chord joining the corresponding points. One may check all the formulas (51) through this model. Thus

$$\left. \begin{aligned} d(\pi_0, \pi) &= (AC) \\ d(-\pi_0, -\pi) &= (BF) \end{aligned} \right| \xrightarrow{(AC) = (BF)} d(\pi_0, \pi) = d(-\pi_0, -\pi) = 2 \left| \sin\frac{\theta}{2} \right| ,$$

$$\left. \begin{aligned} d(-\pi_0, \pi) &= (AB) \\ d(\pi_0, -\pi) &= (CF) \end{aligned} \right| \xrightarrow{(AB) = (CF)} d(-\pi_0, \pi) = d(\pi_0, -\pi) = 2 \left| \cos\frac{\theta}{2} \right| ,$$

$$\left. \begin{aligned} d(\pi_0, -\pi_0) &= (BC) = 2 \\ d(\pi, -\pi) &= (AF) = 2 \end{aligned} \right| \Longrightarrow d(\pi_0, -\pi_0) = d(\pi, -\pi) = 2,$$

and obviously

$$d(\pi_0, \pi_0) = d(\pi, \pi) = 0$$

Remark. One thing that should be stressed here is the linear representation of the "distances" between the faces of planes. It should have been noticed that it is not necessary to inverse the circular functions involved, and express the "distances" in terms of arc.

The two different "distances" which can be defined between the faces of two planes are not independent from each other. They are related through the relation

$$d^2(\pi_0, \pi) + d^2(-\pi_0, \pi) = (AC)^2 + (AB)^2 = 4 \qquad (53)$$

Having the distances between faces of two planes, naturally springs up the question as to whether there exists a way to treat each plane as a unit and find a unique "distance" between two such units. That "distance" has to be a Euclidean metric and consequently a projective metric (see section 2.24). The above-employed representation for the "distances" between faces of two planes (Fig. 10) shows so clearly the existence and uniqueness in magnitude of such a "distance", that it hardly needs any further explanation. Thus the "distance" between the planes π and π_0 is given by the distance (AN) of A (face $+\pi$) from the hypotenuse (BC), (trace of the plane π_0 on the plane of the cross-section). As one checks immediately the distance of A from (BC) is equal to the distance* of F (face $-\pi$) from the same line (BC), as well as with the distances of B (face $-\pi_0$) and C (face $+\pi_0$) from the line (AF), (trace of the plane π on the plane of the cross-section). Therefore, the "distance" between π_0 and π is uniquely defined. Let it be called $D(\pi_0, \pi)$. It remains to be proved that $D(\pi_0, \pi)$ is an Euclidean metric. To this end, the geometrical meaning and the connection of $D(\pi_0, \pi)$ with the "distances" $d(\pi_0, \pi)$ and $d(-\pi_0, \pi)$ will be found as intermediate results. From Fig. 10 one obtains

area (ABC) = $\frac{1}{2}$ (AC) (AB) = $\frac{1}{2}$ (BC) (AN).

$$= \frac{1}{2} \cdot 2 \cdot D(\pi_0, \pi) = D(\pi_0, \pi).$$

or, by recalling (52) and (51)

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*What has been said for A could be repeated for D. Because of the antisymmetry of the cross-section the discussion is given on the one half of the figure only.

$$D(\pi_o,\pi) = \tfrac{1}{2}\,(2\,\left|\sin\tfrac{\theta}{2}\right|)\,(2\,\left|\cos\tfrac{\theta}{2}\right|) = \left|\sin\theta\right|.$$

Thus

$$D(\pi_o,\pi) = \left|\sin\theta\right|,\tag{54}$$

and also

$$D(\pi_o,\pi) = \text{area (ABC)} = \tfrac{1}{2}\,d(\pi_o,\pi)\cdot d(_{-\pi_o},\pi).\tag{55}$$

The extreme values of $D(\pi_o,\pi)$ will be given here without any recourse to the properties of $\sin\theta$. From (53) one concludes immediately that since the sum of $d^2(\pi_o,\pi)$ and $d^2(_{-\pi_o},\pi)$ equals constantly to 4, their product becomes maximum when $d^2(\pi_o,\pi) = d^2(_{-\pi_o},\pi) = 2$. Thus the maximum of $D^2(\pi_o,\pi) = \dfrac{1}{4}\,d^2(\pi_o,\pi)\cdot d^2(_{-\pi_o},\pi)$ is

$$\max\,[D^2(\pi_o,\pi)] = 1.$$

Since $D(\pi_o,\pi)$ is taken positive by definition, it can be written

$$\max\,[D(\pi_o,\pi)] = 1 \text{ or } D(\pi_o,\pi) \leqq 1,\tag{56}$$

that verifies the known result $\left|\sin\theta\right| \leq 1$. One can verify all the known properties of $\sin\theta$.

In order to be shown that $D(\pi_o\pi)$ is an Euclidean metric it is sufficient to recall the relation (55), i.e.,

$$D^2(\pi_o,\pi) = \frac{1}{4}\,d^2(\pi_o,\pi)\cdot d^2(_{-\pi_o},\pi)$$

where $d^2(\pi_o,\pi)$ and $d^2(_{-\pi_o},\pi)$ are Euclidean metrics, as it is known from the preceding discussion. But it is worthwhile to put $D^2(\pi_o,\pi)$ in the general form of an Euclidean metric (see section 2.24, relation (18)). One may proceed as follows:

$$D^2(\pi_o,\pi) = \frac{1}{4}\,d^2(\pi_o,\pi)\cdot d^2(_{-\pi_o},\pi) = \frac{1}{4}\,2(1-\cos\theta)\,2\,(1+\cos\theta)$$

$$= 1-\cos^2\theta$$

$$= 1-(\frac{\alpha_o\alpha + \beta_o\beta + \gamma_o\gamma}{\rho_o\rho})^2$$

$$= \frac{(\beta_0 \gamma - \beta \gamma_0)^2 + (\alpha_0 \gamma - \alpha \cdot \gamma_0)^2 + (\alpha_0 \beta - \beta_0 \alpha)^2}{\rho_0^2 \, \rho^2}$$

or

$$D^2 (\pi_0 , \pi) = \begin{vmatrix} \dfrac{\beta_0}{\rho_0} & \dfrac{\gamma_0}{\rho_0} \\[2mm] \dfrac{\beta}{\rho} & \dfrac{\gamma}{\rho} \end{vmatrix}^2 + \begin{vmatrix} \dfrac{\gamma_0}{\rho_0} & \dfrac{\alpha_0}{\rho_0} \\[2mm] \dfrac{\gamma}{\rho} & \dfrac{\alpha}{\rho} \end{vmatrix}^2 + \begin{vmatrix} \dfrac{\alpha_0}{\rho_0} & \dfrac{\beta_0}{\rho_0} \\[2mm] \dfrac{\alpha}{\rho} & \dfrac{\beta}{\rho} \end{vmatrix}^2 \qquad (57)$$

The last expression may be written

$$D^2 (\pi_0 , \pi) = U^T U \qquad (58)$$

where

$$U^T = \begin{bmatrix} \begin{vmatrix} \dfrac{\beta_0}{\rho_0} & \dfrac{\gamma_0}{\rho_0} \\[2mm] \dfrac{f}{\rho} & \dfrac{\gamma}{\rho} \end{vmatrix} & ; & \begin{vmatrix} \dfrac{\gamma_0}{\rho_0} & \dfrac{\alpha_0}{\rho_0} \\[2mm] \dfrac{\gamma}{\rho} & \dfrac{\alpha}{\rho} \end{vmatrix} & ; & \begin{vmatrix} \dfrac{\alpha_0}{\rho_0} & \dfrac{\beta_0}{\rho_0} \\[2mm] \dfrac{\alpha}{\rho} & \dfrac{\beta}{\rho} \end{vmatrix} \end{bmatrix}^T \qquad (59)$$

This shows that $D^2 (\pi_0 , \pi)$ is a special case of the general expression (18) for a Euclidean metric between points (see relation (32)). Thus, the metric $D(\pi_0 , \pi)$ will be called <u>dual metric</u> of the usual Euclidean metric. As it was proved, $D(\pi_0 , \pi)$ renders the shortest "distance" between the planes π_0 and π (considered as units), in the same way as $d(P_0 , P)$ renders the shortest distance between the points P_0 and P.

The above expression (57) of $D^2 (\pi_0 , \pi)$, is in terms of plane coordinates actually normalized plane coordinates. Two more analytic expressions of $D^2 (\pi_0 \pi)$ very useful for the adjustment problem will be given here. One is in line coordinates (directional cosines) while the other is in line and plane coordinates. For the first, consider the lines (ξ) of π_0 and (ξ) of the π perpendicular to the intersection line (ℓ) of π_0 and π. Then relation (44) yields:

$$D^2 (\pi_0 , \pi) = \sin^2 \theta = \begin{vmatrix} b_\xi & c_\xi \\ b_\zeta & c_\zeta \end{vmatrix}^2 + \begin{vmatrix} c_\xi & a_\xi \\ c_\zeta & a_\zeta \end{vmatrix}^2 + \begin{vmatrix} a_\xi & b_\xi \\ a_\zeta & b_\zeta \end{vmatrix}^2 \qquad (60)$$

For the second, consider again the line (ξ) of π_0 (the same as above), and the vector $[\alpha, \beta, \gamma]$, (which is perpendicular to π). Then

$$\cos(90° -\theta) = a_\xi \cdot \frac{\alpha}{\rho} + b_\xi \frac{\beta}{\rho} + c_\xi \frac{\gamma}{\rho} \quad (\rho = [\alpha^2 + \beta^2 + \gamma^2]^{\frac{1}{2}}).$$

$$D(\pi_o, \pi) = \left|\sin\theta\right| = \frac{a_\xi \alpha + b_\xi \beta + c_\xi \gamma}{\rho}. \tag{61}$$

Recapitulating the results yielded by the above discussion about the "distance" between two planes one has the following:

(i)   Given the equation $\alpha x_1 + \beta x_2 + \gamma x_3 + \delta x_4 = 0$ (of a plane π (in) homogeneous coordinates) or class $[\pi]$ of planes is determined, whose representation vector is $[\lambda\alpha, \lambda\beta, \lambda\gamma, \lambda\delta]$, where $\lambda \neq 0$. For $\lambda > 0$ the subclass $[+\pi]$ is obtained which is determined by one of the faces of the plane π; namely, the face $+\pi \equiv [\alpha, \beta, \gamma]$. For $\lambda < 0$ the subclass $[-\pi]$ is obtained which is determined by the other face of π, namely, the face $-\pi \equiv [-\alpha, -\beta, -\gamma]$.

(ii)   Given the equations

$$\pi_o: \quad \alpha_o x_1 + \beta_o x_2 + \gamma_o x_3 + \delta_o x_4 = 0$$

and

$$\pi: \quad \alpha x_1 + \beta x_2 + \gamma x_3 + \delta x_4 = 0$$

of two planes π_o and π, two classes $[\pi_o]$ and $[\pi]$ are determined with representation vectors $[\lambda\alpha_o, \lambda\beta_o, \lambda\gamma_o, \lambda\delta_o]$ and $[\mu\alpha, \mu\beta, \mu\gamma, \mu\delta]$ respectively, $(\lambda, \mu \neq 0)$. If

$$\frac{\alpha_o \alpha + \beta_o \beta + \gamma_o \gamma}{\sqrt{\alpha_o^2 + \beta_o^2 + \gamma_o^2} \cdot \sqrt{\alpha^2 + \beta^2 + \gamma^2}} = \cos\left(\left[\begin{matrix}\alpha_o \\ \beta_o \\ \gamma_o\end{matrix}\right], \left[\begin{matrix}\alpha \\ \beta \\ \gamma\end{matrix}\right] \right) = \cos\theta,$$

and

$d(+\pi_o, +\pi)$ the "distance" between any two members of the subclasses $[+\pi_o]$ and $[+\pi]$, i.e., $\lambda, \mu > 0$, (with analogous definitions for the "distances" $d(-\pi_o, -\pi)$, $d(-\pi_o, +\pi)$, $d(+\pi_o, -\pi)$, etc.),

then

$$d(+\pi_o, +\pi) = d(-\pi_o, -\pi) = \sqrt{2(1-\cos\theta)} = 2\left|\sin\frac{\theta}{2}\right|$$

$$d(-\pi_o, +\pi) = d(+\pi_o, -\pi) = \sqrt{2(1+\cos\theta)} = 2\left|\cos\frac{\theta}{2}\right|$$

$$d(+\pi_o, -\pi_o) = d(+\pi, -\pi) = 2$$

$$d(+\pi_o, +\pi_o) = d(+\pi, +\pi) = 0$$

These are Euclidean metrics, and express "distances" between faces of two planes.

(iii) There exists a uniquely defined Euclidean metric, $D(\pi_o, \pi)$, which expresses the shortest "distance" between two planes π_o and π. The following expressions for this metric are equivalent:

1/ $D(\pi_o, \pi) = \left| \sin\theta \right|$,

2/ $D(\pi_o, \pi) = \frac{1}{2} \cdot d(\pi_o, \pi) \cdot d(-\pi_o, \pi)$,

3/ $D(\pi_o, \pi) = \dfrac{\left[\begin{vmatrix} \beta_o & \gamma_o \\ \beta & \gamma \end{vmatrix}^2 + \begin{vmatrix} \gamma_o & \alpha_o \\ \gamma & \alpha \end{vmatrix}^2 + \begin{vmatrix} \alpha_o & \beta_o \\ \alpha & \beta \end{vmatrix}^2 \right]^{\frac{1}{2}}}{[\alpha_o^2 + \beta_o^2 + \gamma_o^2]^{\frac{1}{2}} [\alpha^2 + \beta^2 + \gamma^2]^{\frac{1}{2}}}$,

4/ $D(\pi_o, \pi) = \left[\begin{vmatrix} b_\xi & c_\xi \\ b_\zeta & c_\zeta \end{vmatrix}^2 + \begin{vmatrix} c_\xi & a_\xi \\ c_\zeta & a_\zeta \end{vmatrix}^2 + \begin{vmatrix} a_\xi & b_\xi \\ a_\zeta & b_\zeta \end{vmatrix}^2 \right]^{\frac{1}{2}}$,

5/ $D(\pi_o, \pi) = \dfrac{a_\xi \alpha + b_\xi \beta + c_\xi \gamma}{\sqrt{\alpha^2 + \beta^2 + \gamma^2}}$,

where $[a_\xi, b_\xi, c_\xi]$ and $[a_\zeta, b_\zeta, c_\zeta]$ are the direction cosines of the lines (ξ) and (ζ), which lie in the planes π_o and π respectively and are perpendicular to the intersection line (ℓ).

Among all these expressions the last one is of particular interest for the adjustment problem (see section 2.22), and it will turn out to be very convenient. As a matter of fact, this is a linear expression with respect to a_ξ, b_ξ, c_ξ and $\frac{\alpha}{\rho}, \frac{\beta}{\rho}, \frac{\gamma}{\rho}$, and fits the adjustment problem as it was formulated in section 2.22. Thus the required expression for the regression model has been found. The following subsection is devoted to the statistical analysis of this model.

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2.26 Statistical Analysis of the Adjustment Problem

The geometrical analysis of the optical observations mode (see section 2.1) revealed the type of the statistical model, namely, a linear regression model. Although the enunciation of the adjustment problem (i.e., what is the least-squares line through a set of planes) was crystalline about the linearity of the problem, the analytical expression of the model was not so obvious. Being convinced of the linearity of the problem, one was expecting a linear analytical expression. Duality principle offered the means to find out the suitable analytical expression for the linear regression model. The fact that one knows the statistical model does not discharge him from being obligated to discuss it stochastically and to appraise the validity of the statistical assumptions which accompany the model, under the conditions of the problem. This subsection aims at a general statistical treatment of the problem.

2.261 Statistical Model

Two ground stations P_1 and P_2 are carrying out simultaneous optical observations of a set of satellite position points $Q_1, Q_2 \ldots, Q_p$. The observed quantities by station P_i at the event $[E_j, Q_j \mid t_j]$ (see section 1) are equivalent to the direction cosines $[a_{ij}, b_{ij}, c_{ij}]$ of the direction $\vec{P_i Q_j}$. At each event, two directions $\vec{P_1 Q_j}$ and $\vec{P_2 Q_j}$ are determined, which are not independent from each other, for they meet at Q_j. These two directions intersecting at Q_j, determine a plane π_j which is defined as passing from Q_j and being parallel to the directions $P_1 Q_j$ and $P_2 Q_j$ (see section 2.1). Thus, there is a set of planes π_j ($j = 1, 2, \ldots, p$) independently determined from each other. One might think that since the planes π_j are passing through the line $(P_1 P_2)$, they could be considered as members of a pencil of planes with axis, the line $(P_1 P_2)$. But such a thing would kill the problem statistically, for the planes, so defined, are not independent from each other any more. Further discussion on this is postponed until later. These planes determine $C_2^p = p(p-1)/2$ independent

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intersection lines.   The line $(P_1 P_2)$ is considered as the fitting straight line to the set of the intersection lines, or that is the same as the least-squares straight line through the set of planes π_j (j = 1, 2, ...,p).

Let (ξ) be the line $(P_1 P_2)$ with direction cosines $[a_\xi, b_\xi, c_\xi]$.   Then according to the discussion in section 2.22, it corresponds to the line for which the following least squares optimum criterion is satisfied.

$$\sum_{j=1}^{p} D^2(\pi_{oj}, \pi_j) = \text{minimum}, \qquad (62)$$

$$\pi_j : \alpha_j x_1 + \beta_j x_2 + \gamma_j x_3 + \delta_j x_4 = 0$$

Fig. 11
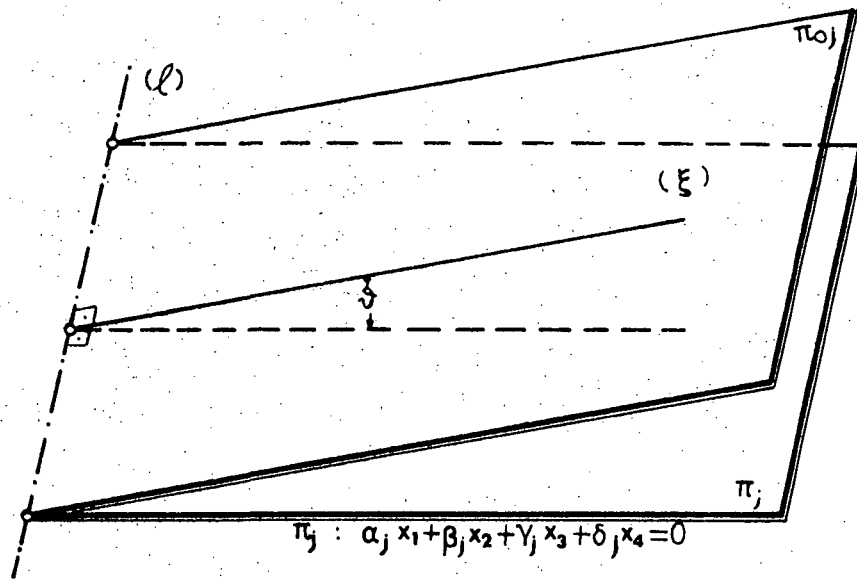
where $D(\pi_{oj}, \pi_j)$ (see section 2.25) is given by (61), i.e.,

$$D(\pi_{oj}, \pi_j) = \frac{\alpha_j}{\rho_j} a_\xi + \frac{\beta_j}{\rho_j} b_\xi + \frac{\gamma_j}{\rho_j} c_\xi ,$$

$$\rho_j = [\alpha_j^2 + \beta_j^2 + \gamma_j^2]^{\frac{1}{2}} .$$

One may write

$$D_j = D(\xi, \pi_j) = D(\pi_{oj}, \pi_j) = \alpha_{Nj} a_\xi + \beta_{Nj} b_\xi + \gamma_{Nj} c_\xi , \qquad (63)$$

where

$$\alpha_{Nj} = \frac{\alpha_j}{\rho_j} \,, \quad \beta_{Nj} = \frac{\beta_j}{\rho_j} \,, \quad \gamma_{Nj} = \frac{\gamma_j}{\rho_j} \,, \quad \delta_{Nj} = \frac{\delta_j}{\rho_j} \,, \tag{64}$$

are the normalized coordinates of π_j.

If (63) is written in the form

$$D_j = \frac{\alpha_{Nj} a_\xi + \beta_{Nj} b_\xi + \gamma_{Nj} c_\xi + 0}{[a_\xi^2 + b_\xi^2 + c_\xi^2]^{\frac{1}{2}}} \,, \quad (j = 1, \ldots, p) \tag{65}$$

where as is known

$$[a_\xi^2 + b_\xi^2 + c_\xi^2] = 1 \,, \tag{66}$$

then one recognized immediately the formula for the distance of a point with coordinates $[\alpha_{Nj}, \beta_{Nj}, \gamma_{Nj}]$ from the plane, the perpendicular vector of which is $[a_\xi, b_\xi, c_\xi]$. This is very remarkable because the problem in question is thus reduced to a very well-known case, namely: find the least-squares plane through a set of points. Hamilton's presentation of the least squares solution to this problem will be adopted here [18, 19].

Let

$$\bar{m} = [a_\xi, b_\xi, c_\xi], \tag{67}$$

and

$$\bar{x}_j = [\alpha_{Nj}, \beta_{Nj}, \gamma_{Nj}]^T \,, \quad (j = 1, 2, \ldots, p) \tag{68}$$

be the coordinate vectors of the "plane" to be fitted and a typical observed "point"* respectively. Then (65) and (66) may be written

$$D_j = \frac{\bar{m} \bar{x}_j + 0}{[\bar{m} \bar{m}^T]^{\frac{1}{2}}} = \bar{m} \bar{x}_j \,, \quad (j = 1, \ldots, p) \,, \tag{69}$$

$$\bar{m} \bar{m}^T = 1 \tag{70}$$

---

*The quotation marks are justified by the fact that the notions point and plane are symbolic there.

The equations (69) are the so-called observation equations while (70) is an absolute constraint. As far as statistical assumptions go, it is assumed that α_{Nj}, β_{Nj} and γ_{Nj} are independent normal random variables. Before proceeding to the least squares solution, some comments on the statistical model are set forth.

2.262 Comments on the Statistical Model

It certainly has been noticed that the "observed quantities" in the above adjustment are the coordinates α_{Nj}, β_{Nj}, γ_{Nj} of the plane π_j. One might wonder how far away these quantities are from the quantities which are actually recorded by the observer. In Principle, the quantity which enters the adjustment as observed, represents the observational unit, so to say, that is the event which is repeated during the observations and which may be considered independent from the preceding and following one. The question now is, what is the observational unit in the optical observations mode? The "instrument" of these observations is not each one of the two instruments at the observing stations, but both of them, and the "observed quantity" is not each one of the two directions, but both of them together as a unit. That unit is nothing else but the plane defined by these two directions which, by fact, meet at the satellite position point. Obviously, this is the observational unit, or the event, if one wishes, which is repeated during the observations and may be considered independent from the other such events.

Therefore, the quantities which enter the adjustment as observed must represent the planes which are defined independently from one another and each by the corresponding simultaneous directions. Thus, the above adjustment cannot be simplified further without violating basic statistical principles about adjustments. But there are many ways to represent one and the same plane. This matter was discussed in section 2.1, where it was concluded that the only statistically permissible representation of the plane π_j (see Fig. 12) is from the two

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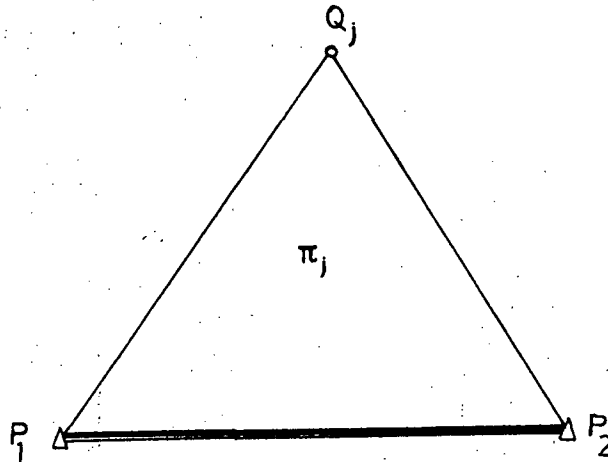
Fig. 12

intersecting directions $\vec{P_1 Q_j}$ and $\vec{P_2 Q_j}$. If one considers the planes π_j, forming a pencil of planes with axis the line ($P_1 P_2$), then the independence of these planes disappears, and a mathematical model for which this is the case, does not yield an adjustment of this problem. From the above discussion it became clear that there was no other way to look at this problem, (i.e., two unknown ground stations observing simultaneously directions to a set of satellite position points). That means one must look at ($P_1 P_2$) as the fitting line of a set of lines (all possible inter-sections of the planes π_j). In the following of the present commentary, a mathematical model for the same problem different from the above which appeared in geodetic publications will be discussed briefly [2]. The mathematical model given here there (somehow arbitrarily) is the following (see Fig. 13).

Direction $P_1 Q_j$:

$$X_j - x_i - r_{1j} a_{1j} = 0$$
$$Y_j - y_1 - r_{1j} b_{1j} = 0$$
$$Z_j - z_1 - r_{1j} c_{1j} = 0$$

Direction P_2Q_j:

$$X_j - x_2 - r_{2j}a_{2j} = 0$$

$$Y_j - y_2 - r_{2j}b_{2j} = 0$$

$$Z_j - z_2 - r_{2j}c_{2j} = 0$$
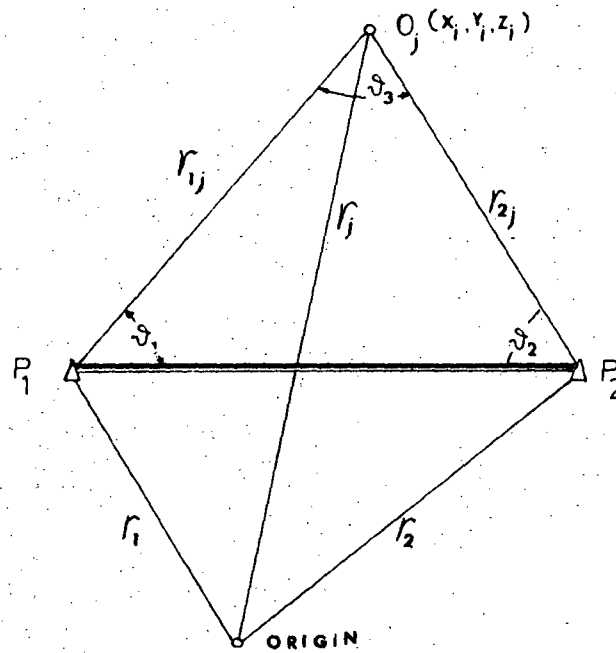
Fig. 13

There is no question that they are necessary, but as it turns out, they are not sufficient for this problem. One may observe the following points.

After elimination of the satellite coordinates, the above conditions are reduced to three. These three resultant equations have the form

$$x_1 - x_2 + r_{1j}a_{1j} - r_{2j}a_{2j} = 0$$

(only the x-component is written for brevity).

One might think that the presence of the coordinates and the distances in the above formulas is intrinsic. But this is not the case. As a matter of fact, the distances r_{1j} and r_{2j} are unknown and they become known

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after the introduction of the length r_{12}, through the relations

$$r_{1j} = r_{12} \frac{\sin \theta_2}{\sin \theta_3} \,,$$

and

$$r_{2j} = r_{12} \frac{\sin \theta_1}{\sin \theta_3} \,.$$

Then

$$\frac{x_1 - x_2}{r_{12}} + \frac{\sin \theta_2}{\sin \theta_3} a_{1j} - \frac{\sin \theta_1}{\sin \theta_3} a_{2j} = 0$$

or

$$(\sin \theta_3)\, a_{12} + (\sin \theta_2)\, a_{1j} - (\sin \theta_1)\, a_{2j} = 0.$$

The last one is independent from the coordinates of the points P_1 and P_2.

Another thing which must be pointed out is that the directions $\vec{P_1Q_j}$ and $\vec{P_2Q_j}$ are introduced independently, while they are not independent for they intersect at Q_j. This dependence is expressed by the equation of the plane through Q_j and parallel to $\vec{P_1Q_j}$ and $\vec{P_2Q_j}$, which is not mentioned anywhere there. These two points show that the above model does not give the required adjustment. It is not easy to visualize what it does. It seems that after such an adjustment each triple $[\vec{P_1P_2}, \vec{P_1Q_j}, \vec{P_2Q_j}]$ is parallel to a plane, but there is no connection between these planes. That model cannot be identified with any of the known statistical models. It belongs to the category of the so-called generalized least squares. There appears to be no objection against the generalized least squares, provided the employed conditions are proved to be not only necessary but sufficient too; sufficient in the sense that they define the configuration uniquely. Unless the sufficiency of the conditions is proved, the results are very precarious. The fact that there is no rigorous statistical foundation for the generalized least squares is an additional reason for some people to keep their reservations with this method. In the last analysis, why should one have recourse to an unconvincing generalized least squares solution, while he has a clearer, fully convincing solution of the adjustment problem ?

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2.263 Least-Squares Solution

Equations (69) and (70) constitute the mathematical model of the problem. The least-squares solution to this problem is very known. As a matter of fact there are more than one approaches to the solution of this problem [17, 18].

Since this mathematical model is very common in statistics there exists a vast bibliography about its statistical analysis (computing techniques for estimation, hypothesis testing, etc.) which is very beneficial for the geodetic problem. Once the problem of the optical observations mode is reduced to that one above, the task of the present work in this direction has been accomplished. Nevertheless for completion purposes, Hamilton's method of solution will be given below.

Hamilton's solution to the problem of fitting a "plane" through a set of "points".

The following matrix notation is introduced

$$\bar{D} = (D_1, D_2, \ldots, D_p), \quad (1 \times p) \tag{71}$$

$$\bar{X} = (\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_p), \quad (3 \times p) \tag{72}$$

$\bar{M}_x$ (3p × 3p)   variance-covariance matrix of $\bar{X}$

$\bar{M}_d$ (p × p)   variance-covariance matrix of $\bar{D}$

$\bar{W} = \bar{M}_d^{-1}$ weight matrix of $\bar{D}$ [to be discussed later]

Point estimation.

Proceeding formally one obtains the function to be minimized, i.e.,

$$\Phi = \bar{D} \, \bar{W} \, \bar{D}^T - k(\bar{m} \bar{m}^T - 1), \tag{73}$$

where k is a Lagrange multiplier for the constraint $\bar{m} \bar{m}^T = 1$. From (69) and (70) one has

$$\bar{D} = (\bar{m} \bar{x}_1, \bar{m} \bar{x}_2, \ldots, \bar{m} \bar{x}_p) = \bar{m} \bar{X} \tag{74}$$

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Then (73) may be written

$$\Phi = \overline{m}(\overline{X}\,\overline{W}\,\overline{X}^{\mathsf{T}})\,\overline{m}^{\mathsf{T}} - k(\overline{m}\,\overline{m}^{\mathsf{T}} - 1).$$

The least-squares estimate of the unknown vector $\overline{m}$ is given as it is known by the solution of the following system:

$$\frac{\partial\Phi}{\partial\overline{m}} = 0$$

or

$$(\overline{X}\,\overline{W}\,\overline{X}^{\mathsf{T}} - kI_3)\,\overline{m}^{\mathsf{T}} = (\overline{C} - kI_3)\,\overline{m} = 0 \qquad (75)$$

But the last equation shows that $\overline{m}^{\mathsf{T}}$ is an eigenvector of $\overline{C}$ corresponding to the eigenvalue k. Since $\overline{C}$ is a symmetric positive definite matrix it has three real eigenvalues, and let them be in the following order

$$\lambda_1 \leq \lambda_2 \leq \lambda_3 \ .$$

Then $\overline{m}^{\mathsf{T}}$ corresponds to $k = \min \lambda = \lambda_1$. By some standard numerical procedure one may obtain the minimum eigenvalue. The eigenvalues of $\overline{C}^{-1}$ are the inverses of those of $\overline{C}$, and the maximum eigenvalue of $\overline{C}^{-1}$ corresponds to the minimum of $\overline{C}$. Since the numerical procedure to find the eigenvalues starts with the maximum eigenvalue one is looking for the maximum eigenvalue of $\overline{C}^{-1}$, whose the inverse is the minimum of $\overline{C}$. Given the fact that $\overline{C} = \overline{X}\,\overline{W}\,\overline{X}^{\mathsf{T}}$ is a (3 x 3) matrix there is not much numerical trouble involved in these manipulations.

Nature of the Weight Matrix W.

In this problem the variance covariance matrix of $\overline{X}$, i.e. $\overline{M}_x$ may be considered diagonal with (3 x 3) diagonal blocks. That because, the planes π_j, (j = 1, 2, . . . , p) may be considered independent. Nevertheless in this general treatment, a full variance-covariance matrix $\overline{M}_x$ is taken and let its typical element be denoted by $\overline{M}_x^{i\,j}$, i.e.,

$$\overline{M}_x = \begin{bmatrix} \overline{M}_x^{1,1} & \cdots & \overline{M}_x^{1,p} \\ \vdots & & \vdots \\ \overline{M}_x^{p,1} & \cdots & \overline{M}_x^{p,p} \end{bmatrix}, \qquad \overline{M}_x^{i,j} \text{ is (3 x 3)} \qquad (76)$$

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Equation (74) may be written

$$
\overline{D}^\mathsf{T} = \begin{bmatrix} {}_1\overline{m}_3 & {}_1 0_3 & \cdots & {}_1 0_3 \\ {}_1 0_3 & {}_1\overline{m}_3 & \cdots & {}_1 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & {}_1\overline{m}_3 \end{bmatrix} \cdot \begin{bmatrix} \overline{x}_1 \\ \overline{x}_2 \\ \overline{x}_p \end{bmatrix} = {}_p\overline{\mathfrak{M}}_{3p} \; {}_{3p}\overline{X}_1 \tag{77}
$$

Then the formula for the propagation of error yields

$$
\overline{W}^{-1} = \overline{M}_d = \overline{\mathfrak{M}} \; \overline{M}_x \overline{\mathfrak{M}}^\mathsf{T} \tag{78}
$$

The typical element of $\overline{M}_d$ is

$$
\overline{M}_d^{ij} = \overline{m} \; \overline{M}_x^{ij} \; \overline{m}^\mathsf{T} \tag{79}
$$

If $\overline{M}_x$ is diagonal, then $\overline{M}_x^{i,j} = 0$ for $i \neq j$ and $\overline{M}_d$ is diagonal too, i.e.,

$$
\overline{M}_d = \begin{bmatrix} (\overline{m} \, \overline{M}_x^{1,1} \, \overline{m}^\mathsf{T}) & 0 & \cdots \cdots & 0 \\ 0 & (\overline{m} \, \overline{M}^{2,2} \, \overline{m}^\mathsf{T}) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & (\overline{m} \, \overline{M}^{p,p} \, \overline{m}^\mathsf{T}) \end{bmatrix} \tag{80}
$$

or

$$
\overline{W} = \overline{M}_d^{-1} = \begin{bmatrix} \dfrac{1}{\overline{m} \, \overline{M}_x^{1,1} \, \overline{m}^\mathsf{T}} & 0 & \cdots \cdots & 0 \\ 0 & \dfrac{1}{\overline{m} \, \overline{M}_x^{2,2} \, \overline{m}^\mathsf{T}} & \cdots & 0 \\ \vdots & & \ddots & \\ 0 & 0 & \cdots & \dfrac{1}{\overline{m} \, \overline{M}_x^{p,p} \, \overline{m}^\mathsf{T}} \end{bmatrix} \tag{81}
$$

Since the weight matrix $\overline{W}$ is a function of $\overline{m}$ the equation (75) is non-linear in $\overline{m}$. Therefore an iterative procedure is followed. One assumes a vector $\overline{m}$, calculates the weights from (78), derives a new value for $\overline{m}$ from (75) and then recycles. In cases of two or three dimensions, like the present one, the process converges in no more than two iterations.

Remark. Because the weights depend on $\overline{m}$ and an iterative procedure is used it is important to note that the minimum eigenvalue of $\overline{C}$ does not necessarily correspond to the best solution. When the solutions to the eigenvalue equation (75) have been found, the quantity $\overline{D}\,\overline{W}\,\overline{D}^{\mathsf{T}}$ should be computed for each eigenvalue. If the ellipsoids of error for the individual points are very eccentric, it frequently happens that the minimum $\overline{D}\,\overline{W}\overline{D}^{\mathsf{T}}$ does not correspond to the minimum eigenvalue [20]. This is particularly likely to happen if the initial choice of $\overline{m}$ for the iteration is far from the correct one.

Covariance matrix of $\overline{m}\,(\,\overline{M_m}\,)$.

From (74) one obtains

$$\overline{D}^{\mathsf{T}} = \overline{X}^{\mathsf{T}}\overline{m}^{\mathsf{T}} \quad \text{or} \quad \overline{X}\,\overline{W}\,\overline{D}^{\mathsf{T}} = \overline{X}\,\overline{W}\,\overline{X}^{\mathsf{T}}\,\overline{m}^{\mathsf{T}} = \overline{C}\,\overline{m}^{\mathsf{T}}$$

and

$$\overline{m}^{\mathsf{T}} = (\overline{C}^{-1}\,\overline{X}\,\overline{W})\,\overline{D}^{\mathsf{T}} \tag{82}$$

By the formula for the propagation of error the last relation yields

$$
\begin{aligned}
\overline{M_m} &= (\overline{C}^{-1}\,\overline{X}\,\overline{W})\,\overline{M}_d\,(\overline{W}\,\overline{X}^{\mathsf{T}}\,\overline{C}^{-1}) \\
&= \overline{C}^{-1}\,\overline{X}\,\overline{W}\,\overline{M}_d\,\overline{M}_d^{-1}\,\overline{X}^{\mathsf{T}}\,(\overline{X}\,\overline{W}\,\overline{X}^{\mathsf{T}})^{-1} \\
&= \overline{C}^{-1}(\overline{X}\,\overline{W}\,\overline{X}^{\mathsf{T}})\,(\overline{X}\,\overline{W}\,\overline{X}^{\mathsf{T}})^{-1}
\end{aligned}
$$

or

$$\overline{M_m} = \overline{C}^{-1}. \tag{83}$$

Testing hypothesis.

Consider the hypothesis $H: \overline{m}_0 = \overline{m}$. Assuming that the distribution of $\overline{D}$ is normal then the quadratic form

$$Q^2 = (\overline{m}_0 - \overline{m})\,\overline{C}(\overline{m}_0 - \overline{m})^{\mathsf{T}} \tag{84}$$

is distributed as $3F_{3,p}$, where F is the well-known variance-ratio distribution and 3 is the number of unknown parameters. If Q^2 is less than the tabular value, the hypothesis is accepted. That means $\overline{m}_0$ may characterize the "plane," which is the line (ξ) in this problem. Since

$$\lim_{p \to \infty} 3F_{3,p} = \chi_3^2$$

for large p one may use χ^2 test instead of F.

In every statistical problem, after the statistical model has been chosen, the goodness of the estimation and the statistical inferences depends upon the weights assigned to the observed quantities. For the present problem, the difficulty is about assigning the weight to the observation vector $\overline{x}_j$. By recalling equations (2) – (5) (see section 2.1), one has

$$\overline{x}_j = [\alpha_{Nj}, \beta_{Nj}, \gamma_{Nj}]^T = \frac{1}{\rho_j}[\alpha_j, \beta_j, \gamma_j]^T$$

or

$$\overline{x}_j = \left[\begin{vmatrix} b_{1j} & c_{1j} \\ b_{2j} & c_{2j} \end{vmatrix}, \begin{vmatrix} c_{1j} & a_{1j} \\ c_{2j} & a_{2j} \end{vmatrix}, \begin{vmatrix} a_{1j} & b_{1j} \\ a_{2j} & b_{2j} \end{vmatrix} \right] \tag{85}$$

where

$$\rho_j = [\alpha_j^2 + \beta_j^2 + \gamma_j^2]^{\frac{1}{2}}.$$

It is interesting to note that ρ_j is the sine of the angle which is formed by the two observed directions at the satellite position point (see relation (44)). If the observed angular quantities are the declination (δ) and the hour angle (h) the following relations hold

$$a_{1j} = \cos\delta_{1j} \cos h_{1j} \qquad\qquad a_{2j} = \cos\delta_{2j} \cos h_{2j}$$

$$b_{1j} = \cos\delta_{1j} \sin h_{1j} \qquad\qquad b_{2j} = \cos\delta_{2j} \sin h_{2j} \tag{86}$$

$$c_{1j} = \sin\delta_{1j} \qquad\qquad\qquad c_{2j} = \sin\delta_{2j}$$

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Then

$$\alpha_j = \begin{vmatrix} b_{1j} & c_{1j} \\ b_{2j} & c_{2j} \end{vmatrix} = \begin{vmatrix} \cos\delta_{1j}\cdot\sin h_{1j}, & \sin\delta_{1j} \\ \cos\delta_{2j}\cdot\sin h_{2j}, & \sin\delta_{2j} \end{vmatrix}, \tag{87}$$

$$\beta_j = -\begin{vmatrix} a_{1j} & c_{1j} \\ a_{2j} & c_{2j} \end{vmatrix} = -\begin{vmatrix} \cos\delta_{1j}\cdot\cos h_{1j}, & \sin\delta_{1j} \\ \cos\delta_{2j}\cdot\cos h_{2j}, & \sin\delta_{2j} \end{vmatrix}, \tag{88}$$

$$\gamma_j = \begin{vmatrix} a_{1j} & b_{1j} \\ a_{2j} & b_{2j} \end{vmatrix} = \begin{vmatrix} \cos\delta_{1j}\cos h_{1j} & \cos\delta_{1j}\sin h_{1j} \\ \cos\delta_{2j}\cos h_{2j} & \cos\delta_{2j}\sin h_{2j} \end{vmatrix} \tag{89}$$

$$\rho_j = \sin(\overrightarrow{P_1 Q_j}, \overline{P_2 Q_j}) = \sin\theta_j \; (\cos^{-1}\theta_j)$$

$$= [1-(a_{1j}a_{2j}+b_{1j}b_{2j}+c_{1j}c_{2j})^2]^{\frac{1}{2}}$$

or

$$\rho_j{}^2 = 1-[\sin\delta_{1j}\sin\delta_{2j}+\cos\delta_{1j}\cos\delta_{2j}\cos(h_{1j}-h_{2j})]^2 \tag{90}$$

The above formulas are not too complicated to compute the errors of

α_{Nj}, β_{Nj}, γ_{Nj}, by using the formula for propagation of error on the linearized

expressions of the above when the errors of δ_{1j}, h_{1j}, δ_{2j}, h_{2j} are given. It

is stressed again that the observational unit of this problem is the plane π_j,

and not the directions $\overrightarrow{P_1 Q_j}$ and $\overrightarrow{P_2 Q_j}$ individually. Hence, the coordinates of

this plane are the "observed quantities" for the adjustment problem and these

are those which must be assigned a weight. As it was inferred above, the plane

π_j is the simplest observational unit and any mathematical model whose observa-

tional unit is simpler than that plane does not adjust the observations of this

problem.

2.3 Determination of the Cartesian Coordinates of the Stations

After the above adjustment one has at its disposition the direction cosines of the line $(\xi) \equiv (P_1 P_2)$. In order to compute the coordinates, one needs the coordinates of either one of the two points and the length. Let x_1, y_1, z_1 be the coordinates of P_1, and S_{12} be the length of the distance $(P_1 P_2)$. Then

$$
\begin{aligned}
x_2 &= a_\xi S_{12} + x_1 \\
y_2 &= b_\xi S_{12} + y_1 \\
z_2 &= c_\xi S_{12} + z_1
\end{aligned}
\qquad (91)
$$

3. THE CASE OF MORE THAN TWO STATIONS

Let $P_1, P_2, \ldots, P_1$ be a set of ground stations. Consider all the pairs of stations from this set, each one of which has observed simultaneously at least one satellite position. It does not matter whether some of the satellite position points take part in more than one pair. Let these pairs be classified
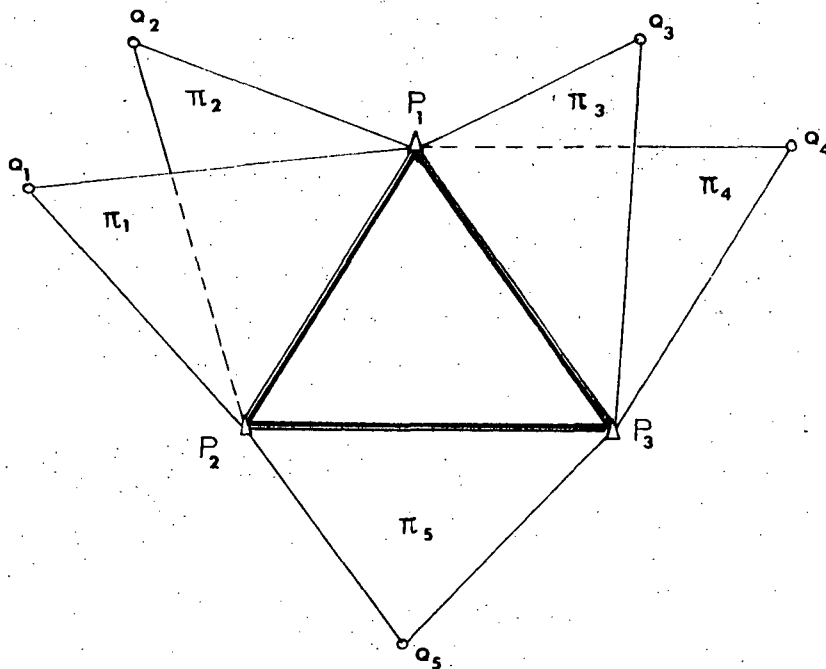

Fig. 14

60

as follows:

A₃ set of station pairs each of which has observed more than two satellite

  position points,

A₂ set of station pairs each of which has observed just two satellite

  position points,

A₁ set of station pairs each of which has observed just one satellite

  position point.

Remark. It is possible, of course, one and the same station, say P_k, to be

found in all these three sets of pairs, but never as it is understood, two and

the same stations can be found in two different sets of pairs.

1st Estimation.

For each station-pair of the set A_3, an adjustment is carried out

which yields the directional cosines of the line of that pair. For each station-

pair of the set A_2, there is no adjustment problem inasmuch as there exists

only one intersection line, which is taken as the line of that station-pair. If,

for example, (P_1, P_2) is a station-pair which has observed two satellite points

Q_1 and Q_2 (see Figure 14), the only intersection line is that of the planes

π_1 and π_2 (see Section 2.1). For the station-pairs of the set A_1 there is no

information during the first estimation. Thus, the first estimation consists

of the direct information provided to the ground stations from the satellite

observations, that is, the directional cosines of the lines of the station-pairs

in A_3 and A_2.

After the first estimation one has additional information for the deter-

mination of the directional cosines of the lines of the station-pairs, which

may be called indirect observations. That is, in a second estimation of the

direction cosines of the line of a station-pair (P_1, P_2) one may consider not

only the planes π_j corresponding to the satellite position points Q_j observed

by the stations P_1 and P_2, but also the planes $\Pi_{12}^i \equiv (P_1 P_2 P_i)(i = 3, 4, \ldots)$,

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provided the directions $\overrightarrow{P_1 P_i}$ and $\overrightarrow{P_2 P_i}$ have been estimated in the first estimation. Consider, for example, the simple configuration of three stations P_1, P_2, P_3 (see Figure 14), where the lines $(P_1 P_2)$, $(P_2 P_3)$ and $(P_3 P_1)$ have known directional cosines from the first estimation. The directions $\overrightarrow{P_1 P_2}$, $\overrightarrow{P_2 P_3}$, $\overrightarrow{P_3 P_1}$ as they result from the first estimation, are not in general coplanar. One might consider here the coplanarity condition of three directions, i.e.,

$$F(a_{ij}, b_{ij}, c_{ij}) = \begin{vmatrix} a_{12} & b_{12} & c_{12} \\ a_{23} & b_{23} & c_{23} \\ a_{31} & b_{31} & c_{31} \end{vmatrix} = 0 , \qquad (92)$$

and adjust the results of the first estimation according to this condition. This way is followed in [1]. But taking the first estimation results and enforcing them to verify condition (92) it is like killing the problem statistically, and abandoning the principles adopted in this treatment for the case of two stations by implicating statistical models for which theories about statistical inferences have dubious validity; the function $F(a_{ij}, b_{ij}, c_{ij})$, $(i, j = 1, 2, 3)$ has nine variables and it is of third degree. Therefore, by principal, the adjustment here will <u>not</u> proceed this way. The natural way to proceed is that of a second estimation, as already quoted above.

2nd Estimation.

The second estimation is similar to the first, and only the number of observations has been increased, by taking into account not only the direct observations (which yielded the first estimation results), but the indirect observations too. Consider a station-pair, say (P_1, P_2), and let the following sets be defined

$$B = [\pi] = [\pi_1, \ldots, \pi_j, \ldots]$$

$$C = [\Pi] = [\Pi_{12}^1 \ldots, \Pi_{12}^1, \ldots] ,$$

where B is the set of planes corresponding to the satellite points Q_j having

been observed by the station-pair (P_1, P_2) and C is the set of planes $\Pi_i \equiv (P_1 P_2 P_i)$ corresponding to the ground stations P_i, such that $\vec{P_1 P_i}$ and $\vec{P_2 P_i}$ have been determined in the first estimation. It is observed that in the second estimation one obtains the direction cosines of a station-pair whenever the set C is not empty, (the set B is not empty by fact). Thus, while one could not estimate the direction cosines of a station-pair in the set A_1 in the first estimation, he might be able to do that in the second one, provided the set C of the station-pair in A_1 is not empty.

Note. One should be aware of the definition of the plane Π_{12}^i. It is defined as the plane which passes through the station P_i (upper index) and it is parallel to the directions $\vec{P_1 P_i}$ and $\vec{P_2 P_i}$.

After the second estimation a third one should follow in which the observations consists of the direct (original) and the indirect ones resulted from the second estimation. Thus, an iterative procedure is established which obviously leads to the coplanarity condition. The iteration steps continue until no significant difference exists between the results of the last step and the one immediately preceding it.

After that the adjustment is complete and for the Cartesian coordinates of the stations one has to introduce the coordinates of one station and the length of the distance between two of the stations whose direction has been determined during the adjustment.

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ERRATA SHEET

Reports of the Department of Geodetic Science, No. 185
The Ohio State University
Emmanuel Tsimis

Page	Line From Top	Bottom	Reads	Corrections
2	15		following.	follows.
7		9	π_{12}^{j}	π_j
7		7	π_j	π_k
9		3	subsection	section
11	17, 18		Insert the following title: between two lines and underline.	<u>Duality in Analytic Geometry</u>
12	3		equations	equation
17	9		y's and x's	y's on x's
18	13		Π_j	π_j
22	10, 11		is not supposed to be given here, but anyway it	Remove phrase.
22	18, 19		to be defined below	Remove phrase.
27	17		geometry geometry	geometry
27		2	$\sum_{i=1}^{3} \begin{vmatrix} x_1^0 & x_1 \\ x_4^0 & x_4 \end{vmatrix} \dfrac{1}{x_4^0 x_4}$	$\sum_{i=1}^{3} \begin{vmatrix} x_1^0 & x_1 \\ x_4^0 & x_4 \end{vmatrix} \dfrac{1}{x_4^0 x_4} $
29		7	That is why the proceeding analysis has been placed before this point.	Remove sentence.
31	7		<u>Case 2</u>	<u>Case 2.</u> Plane-Euclidean Space (E_3^{π}, d).
38		9	The the following	Then the following
39		4	$2 \left[1 - \dfrac{\alpha_0(-\alpha_0)+\beta_0(-\beta_0)+\gamma_0(-\gamma_0)}{\rho_0\rho_0} \right.$	$2\left(1 - \dfrac{\alpha_0(-\alpha_0)+\beta_0(-\beta_0)+\gamma(-\gamma_0)}{\rho_0\rho_0} \right)$
40	2		$d(\pi_0,\pi) = \cdots = \sqrt{2(1-\cos\theta)}$	$d(-\pi_0,\pi) = \cdots = \sqrt{2(1+\cos\theta)}$
42		9	with	to
42		6	an Euclidean	a Euclidean

Page	Line From Top	Line From Bottom	Reads	Corrections
42	Foot	note	D	F
43		9	an Euclidean	a Euclidean
43		5	an Euclidean	a Euclidean
44	7		The corrected relation (59):	$U^\tau = \left[\begin{vmatrix} \dfrac{\beta_0}{\rho_0} & \dfrac{\gamma_0}{\rho_0} \\ \dfrac{\beta}{\rho} & \dfrac{\gamma}{\rho} \end{vmatrix}, \begin{vmatrix} \dfrac{\gamma_0}{\rho_0} & \dfrac{\alpha_0}{\rho_0} \\ \dfrac{\gamma}{\rho} & \dfrac{\alpha}{\rho} \end{vmatrix}, \begin{vmatrix} \dfrac{\alpha_0}{\rho_0} & \dfrac{\beta_0}{\rho_0} \\ \dfrac{\alpha}{\rho} & \dfrac{\beta}{\rho} \end{vmatrix} \right]$
44	9		$\dfrac{f}{\rho}$	$\dfrac{\beta}{\rho}$
44		5	(ξ)	(ζ)
45	6		or class	a class
45	9		π_1	π_J
45		10	$\sqrt{\alpha_0^2 + \beta_0^2 + \gamma_0^2} \cdot \sqrt{\alpha^2 + \beta^2 + \gamma^2}$	$\sqrt{\alpha_0^2 + \beta_0^2 + \gamma_0^2} \sqrt{\alpha^2 + \beta^2 + \gamma^2}$
47		8	$P_1 Q_J$ and $P_2 Q_J$	$\overrightarrow{P_1 Q_J}$ and $\overrightarrow{P_2 Q_J}$
49	8		recognized	recognizes
49	12		Hamilton's	W. C. Hamilton's
50	2, 3		As far as statistical assumptions go, it is assumed that α_{NJ}, β_{NJ}, γ_{NJ} are independent normal random variables.	Remove sentence.
50	10		In Principle	In principle
51		6	here	Remove word.
54	9, 10		Hamilton's	W. C. Hamilton's
58	2		number of unknown	number of the unknown
58	13		The corrected relation (85):	$\bar{x}_J = \dfrac{1}{\rho_J} \left[\begin{vmatrix} b_{1J} & c_{1J} \\ b_{2J} & c_{2J} \end{vmatrix}, \begin{vmatrix} c_{1J} & a_{1J} \\ c_{2J} & a_{2J} \end{vmatrix}, \begin{vmatrix} a_{1J} & b_{1J} \\ a_{2J} & b_{2J} \end{vmatrix} \right]^T$
59	11		$= \sin \theta_J (\cos^{-1} \theta_J)$	$= \sin (\cos^{-1} \theta_J)$
60				Delete Section 2.3